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Problems for convergence properties

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§. 1. Definitions. All spaces are assumed to be T_3 and T_1 topological spaces. A space X is said to be bi-sequential if, whenever $\mathcal{F}$ is a filter in X with a cluster point x, then there exists a countable filter base $\mathcal{H}$ in X which converges to x and whose elements intersect all elements of $\mathcal{F}$. If the definition of bi-sequential space is modified by restricting $\mathcal{F}$ to be a countable filter base, the resulting concept is said to be strongly Fréchet. A space X is said to be Fréchet if $x \in \bar{A}$ for $A \subset X$, then there exists a sequence in A converging to the point x.

Let X be a space. A collection $\mathcal{A}$ of convergent sequences of X is said to be a sheaf in X if all member of $\mathcal{A}$ converge to the same point of X, which is said to be the vertex of the sheaf $\mathcal{A}$. In this paper all sheaves are assumed to be countable. We consider the following four properties of X which were introduced by Arhangel'skii[1,2].

Let $\mathfrak{A}$ be a sheaf in X with the vertex $x \in X$. Then there exists a sequence B converging to x such that:

(α_1) $\quad |A - B| < \aleph_0$ for $A \in \mathfrak{A}$,

(α_2) $\quad |A \cap B| = \aleph_0$ for $A \in \mathfrak{A}$,

(α_3) $\quad |\{A \in \mathfrak{A} : |A \cap B| = \aleph_0\}| = \aleph_0$,

(α_4) $\quad |\{A \in \mathfrak{A} : A \cap B \neq \emptyset\}| = \aleph_0$.

The class of spaces satisfying the property (α_i) for every sheaf $\mathfrak{A}$ and vertex $x \in X$ is denoted by $<\alpha_i>$ for i = 1,2,3,4. We denote by $<\alpha_i\text{-FU}>$ the intersection of the class of Fréchet spaces and the class $<\alpha_i>$ for i = 1,2,3,4.

The following diagram shows the relationship between the above spaces and other spaces.

$$\begin{tikzcd}
 & <\alpha_1> \arrow[r] & <\alpha_2> \arrow[r] & <\alpha_3> \arrow[r] & <\alpha_4> & \text{sequential} \\
\text{1'st countable} \arrow[ur] \arrow[dr] & & & \text{w-space} \arrow[u] & & \\
 & <\alpha_1\text{-FU}> \arrow[r] \arrow[uu] & <\alpha_2\text{-FU}> \arrow[r] \arrow[uu] \arrow[ur, equal] & <\alpha_3\text{-FU}> \arrow[r] & <\alpha_4\text{-FU}> \arrow[r] \arrow[uu] & \text{Fréchet} \arrow[uu] \\
 & \text{bi-sequential} \arrow[urr] & & & \text{strongly Fréchet} \arrow[u, equal] & \\
 & & & & \text{countably compact Fréchet} \arrow[u] &
\end{tikzcd}$$

§. 2. Classification problems.

Problem 2-1. Is there a "naive" countable $<\alpha_1$ -FU>-space which is not first countable?

Problem 2-2. Is there a "naive" $<\alpha_2>$-space which is not an $<\alpha_1>$-space?

Remark. Olson's example [see 5, Introduction] is an $<\alpha_2$-FU>-space, and not first countable, so in every model of set theory it solves either Problem 1 or 2. If we omit "countable" in Problem 1, then we get an example. In fact Σ-product of more than countable number of first countable spaces is such a space [2, 6.16].

Problem 2-3. Is there a "naive" $<\alpha_3>$-space which is not an $<\alpha_2>$-space?

Remark. If we assume (CH), then there exists an $<\alpha_3>$-space which is not an $<\alpha_2>$-space.

§. 3. Product problems for Fréchet spaces.

Let P be a class of spaces. Let $\mathfrak{F}(P) = \{X: X \times Y$ is Fréchet for any $Y \in P\}$.

We use the following notations:

C = the class of compact Fréchet spaces,

CC = the class of countably compact Fréchet spaces,

B = the class of bi-sequential spaces,

S = the class of strongly Fréchet spaces

Problem 3-1. Is $\mathfrak{F}(C) = \mathfrak{F}(CC)$?

Problem 3-2. Give inner characterizations of classes $\mathfrak{F}(C)$, $\mathfrak{F}(CC)$ and $\mathfrak{F}(S)$.

Remark. $<\alpha_3\text{-FU}> \subset \mathfrak{F}(CC) \subset \mathfrak{F}(C) \subset S$ and $B \subset \mathfrak{F}(S) \subset \mathfrak{F}(CC)$. Note that $\mathfrak{F}(B) = S$. If we assume (CH), then $<\alpha_3\text{-FU}> \subsetneqq \mathfrak{F}(CC)$ and $\mathfrak{F}(S) \subsetneqq \mathfrak{F}(CC)$.

Problem 3-3. Is there a "naive" example of $\mathfrak{F}(CC)$-space which is not bi-sequential?

Problem 3-4. Is there a "naive" example of $\mathfrak{F}(CC)$-space which is not an $<\alpha_3\text{-FU}>$-space?

Problem 3-4. Is $B = \mathfrak{F}(S)$?

§. 4. Miscellaneous problems.

Problem 4-1(Arhangel'skii). Is $t(X^2) = t(X)$ for each countably compact regular space X?

Problem 4-2(Arhangel'skii). Give an inner characterization of subsequential spaces (A space is said to be subsequential if it can be embedded in a sequential space.).

Problem 4-3(Gerlits-Nagy). Let X and Y be G-spaces. Is $t(X \times Y) \leq \mathfrak{c}$? A space X is said to be G-space if each countable subspace is first countable and X has countable tightness.

A space is said to be quasi-prime if X is embedded as a closed subset of $\prod_{i\in N} Y_i$, then there exists an $n \varepsilon N$ such that X is embedded in $\prod_{i=1}^{n} Y_i$, where Y_i, $i \varepsilon N$ are arbitrary spaces. For example $N \cup \{p\} (p \varepsilon N^*)$ is quasi-**prime** [3].

Problem 4-3. Let Q be the space of rationals. Is βQ quasi-prime?

Problem 4-4. Is Q quasi-prime?

References

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