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Classifications and Base Enumerations of the Maximal Sets of Three-Valued Logical Functions

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Abstract. Functional completeness theory of P_k involves classifying functions of a closed set of P_k by using all its maximal sets. This also divides all its bases into finite equivalence classes. This paper presents classifications and enumerations of all bases for the set P_3 and all its 18 maximal sets.

1. Introduction. The set of k-valued logical functions, i.e. the union of all the functions $\{f \mid E_k^n \to E_k,$ for $E_k=\{0,1,\ldots,k-1\}$ and $n=0,1,2,\ldots\}$ is denoted by P_k. A subset F of P_k is said to be **closed** if it contains all superpositions of its members (cf. [6, 23]). For closed sets F and H such that $F \subset H$ (proper inclusion), F is **H-maximal** set if there is no closed set G such that $F \subset G \subset H$. A subset X of H is **complete in H** if H is the least closed set containing X. If the number m of H-maximal sets is finite then a subset of functions in H is complete in H if and only if it is not contained in any one H-maximal set (completeness condition)(cf. [6]). Investigations of completeness and related topics, which are usualy called functional completeness problems are directly related to logical circuit design, and they have a wide area of applications in addition to their mathematical importance.

A complete set X in H is called **base of H** if no proper subset of X is complete in H. A set of functions $\{f_1,\ldots,f_s\}$ is called **pivotal in H**, if for each i, $1 \le i \le s$, there exists an H-maximal set H_l which does not contain f_i while all the other functions f_j $(j=1,\ldots,s, j \ne i)$ are elements of H_l (pivotalness condition). From these definitions it follows that a complete pivotal set is a base. The **rank** of a base (pivotal set) is the number of its elements.

We classify the set H of functions into nonempty equivalence classes by using all its maximal sets as indicated below. Then we can discuss the completeness in H in terms of these classes instead of individual functions; if a set is complete, then by replacing a function in the set by any function in the corresponding equivalence class yields another complete set.

The **characteristic vector** of $f \in H$ is $a_1 \dots a_m$, where $a_i=0$ if $f \in H_i$ and $a_i=1$ otherwise($1 \leq i \leq m$). All functions $f \in H$ with the same characteristic vector form a **class of functions**. For a given set $F \subseteq H$ the **class of F** is the set of classes of $f \in F$. The conditions of completeness and pivotalness of F can be conveniently checked by using characteristic vectors corresponding to the class of F.

If we have a complete list of characteristic vectors for nonempty classes of a set, we can enumerate all its bases (pivotal sets). All bases (pivotal sets) with the same class form a **class of bases (pivotal sets)**.

We use the notation of functions preserving a relation to describe H-maximal sets [cf. 23]. An h-ary relation ρ on E_k, $h \geq 1$, is a subset of E_k^h whose elements are written as columns

$$(a_1, \dots, a_h)^T \in \rho \Leftrightarrow (a_{1i}, \dots, a_{hi})^T \in \rho \text{ for all } i,\ 1 \leq i \leq n,$$
$$\text{where } a_j=(a_{j1}, \dots, a_{jn}),\ 1 \leq j \leq h.$$

The relation ρ is written as a matrix whose columns are elements of the relation ρ.

Then set of functions **preserving** $\boldsymbol{\rho}$ (denoted by Pol ρ) is defined by

$$\text{Pol } \rho = \{ f \mid (a_1, \dots, a_h)^T \in \rho \Rightarrow (f(a_1), \dots, f(a_h))^T \in \rho \}.$$

Theorem [6]. P_3 has exactly the following 18 maximal sets:

$T_0=\text{Pol}(0)$, $T_1=\text{Pol}(1)$, $T_2=\text{Pol}(2)$, $T_{01}=\text{Pol}(0\ 1)$, $T_{02}=\text{Pol}(0\ 2)$, $T_{12}=\text{Pol}(1\ 2)$,

$B_0=\text{Pol}\begin{pmatrix} 0&1&2&0&1&0&2 \\ 0&1&2&1&0&2&0 \end{pmatrix}$, $B_1=\text{Pol}\begin{pmatrix} 0&1&2&0&1&1&2 \\ 0&1&2&1&0&2&1 \end{pmatrix}$, $B_2=\text{Pol}\begin{pmatrix} 0&1&2&0&2&1&2 \\ 0&1&2&2&0&2&1 \end{pmatrix}$,

$U_0=\text{Pol}\begin{pmatrix} 0&1&2&1&2 \\ 0&1&2&2&1 \end{pmatrix}$, $U_1=\text{Pol}\begin{pmatrix} 0&1&2&0&2 \\ 0&1&2&2&0 \end{pmatrix}$, $U_2=\text{Pol}\begin{pmatrix} 0&1&2&0&1 \\ 0&1&2&1&0 \end{pmatrix}$,

$M_0=\text{Pol}\begin{pmatrix} 0&1&2&2&2&0 \\ 0&1&2&0&1&1 \end{pmatrix}$, $M_1=\text{Pol}\begin{pmatrix} 0&1&2&0&0&1 \\ 0&1&2&1&2&2 \end{pmatrix}$, $M_2=\text{Pol}\begin{pmatrix} 0&1&2&1&1&2 \\ 0&1&2&2&0&0 \end{pmatrix}$,

$L=\text{Pol}(\{ (a,b,c)^T \in E_3^3 \mid c=2(a+b) \pmod 3 \})$, $S=\text{Pol}\begin{pmatrix} 0&1&2 \\ 1&2&0 \end{pmatrix}$,

$T=\text{Pol}(\{ (a,b,c)^T \in E_3^3 \mid a=b \text{ or } a=c \text{ or } b=c \})$.

2. Classifications of functions. Determination of maximal sets for the set P_k and its closed sets has been subject of investigation in growing number of papers ([20, 5, 6, 21, 22, 12, 2, 3, 10, 11]). Next step, description of classes of functions and classes of bases was done first for the set P_2([5, 4, 8]). First attempt to derive classes of functions of P_3 was done in [13]. This paper also give the notion of pivotal sets as necessary conditions for a set to be base. But, it counted several characteristic vectors twice as different classes, consequently the number of bases reported in [14] was incorrect; this was corrected in [24]. The following table present the number of maximal sets and the number of classes of functions for the sets P_2, P_3 and all P_3-maximal sets. Several classification results exist for some of closed sets of P_k [26, 29, 30, 19].

	P_2	P_3	B_0	M_1	T_0	U_2	T_{01}	T	L	S
maximal sets	5 [20, 5]	18 [6]	7 [10]	13 [12]	12 [10]	13 [10]	15 [10]	5 [10]	5 [2]	2 [3]
classes of functions	15 [5, 4, 8]	406 [13, 24]	54 [15]	88 [25]	253 [17]	383 [27]	607 [28]	6 [16]	10 [16]	4 [16]

3. Enumerations of bases. Two algorithms for the enumeration of bases and pivotal sets are given: [14, 18, 34] and [24, 18, 34]. They are compared in [18, 34].

The numbers of classes of bases and pivotal incomplete sets for the same sets as in the former table are shown in the following two tables. There are several results about maximal rank of a base of P_3 [9, 14] and two proofs that maximal rank of a base of P_3 is 6: computational [14] and theoretical [36].

Classes of bases

rank	P_2 [4, 8]	P_3 [24]	B_0 [15]	M_1 [25]	T_0 [17]	U_2 [27]	T_{01}	T [16]	L [16]	S [16]
1	1	1	-	-	1	1	1	-	-	1
2	17	8265	28	-	4492	4344	12259	-	18	1
3	22	794256	999	1514	234031	680285	2580026	6	6	-
4	2	4612601	2831	40104	552927	7300491	38508259	-	-	-
5	-	810474	724	75209	91377	7627060	53641851	-	-	-
6	-	14124	17	1916	892	944257	7545748	-	-	-
7	-	-	-	1	-	15804	35616	-	-	-
Σ	42	6239721	4599	118744	883720	16572242	102323760	6	24	2

Pivotal incomplete sets

rank	P_2 [26]	P_3	B_0	M_1	T_0	U_2	T_{01}	T	L	S
1	13	404	53	87	251	381	605	5	9	2
2	31	60335	931	3153	21363	57284	147266	10	10	-
3	7	1418970	3678	37946	202689	1594342	6385808	-	-	-
4	-	2677899	2240	96323	149804	5057975	32278690	-	-	-
5	-	176187	168	15087	6595	1911408	18947380	-	-	-
6	-	1368	1	55	8	96464	1198502	-	-	-
7	-	9	-	-	-	240	648	-	-	-
Σ	51	4335172	7071	152651	380710	8718094	58958899	15	19	2

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