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Algorithms related to Diophantine approximations and their applications

by

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§1. On the simple continued fraction expansion

Any irrational number α in the interval $[0,1)$ has a simple continued fraction expansion

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} \tag{1.1}$$

where the digits a_i's are positive integers.

The expansion (1.1) is obtained through the following map $S: [0,1) \circlearrowleft$ such that

$$S\alpha = \frac{1}{\alpha} - \left[\frac{1}{\alpha}\right] . \tag{1.2}$$

Put

$$a(\alpha) = \left[\frac{1}{\alpha}\right]$$

and

$$a_n(\alpha) = a(S^{n-1}\alpha) \qquad n=1,2,\cdots ,$$

then $a_1(\alpha), a_2(\alpha), \cdots$ are just the digits in the expansion (1.1). Accordingly, we call the pair $([0,1), S)$ the algorithm which induces a simple continued fraction expansion.

We define (q_n, p_n) $(=(q_n(\alpha), p_n(\alpha)))$ recursively by

$$\begin{pmatrix} q_0 & q_{-1} \\ p_0 & p_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} = \begin{pmatrix} q_{n-1} & q_{n-2} \\ p_{n-1} & p_{n-2} \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} . \qquad (1.3)$$

Then standard induction arguments show that

$$\alpha = \frac{p_n + S^n\alpha \cdot p_{n-1}}{q_n + S^n\alpha \cdot q_{n-1}} \qquad (1.4)$$

and
$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \ddots + \cfrac{1}{a_n}} \quad \text{and} \quad \frac{q_{n-1}}{q_n} = \cfrac{1}{a_n + \ddots + \cfrac{1}{a_1}}$$

we call (q_n, p_n) n - th principal convergent of α .

We know several metrical results concering the approximation of α with (q_n, p_n).

Metrical theorems

(1) $\lim_{n\to\infty} \frac{1}{n} \log\left|\alpha - \frac{p_n}{q_n}\right| = - \frac{\pi^2}{6\log 2}$

(2) $\lim_{n\to\infty} \frac{1}{n} \log q_n = \frac{\pi^2}{12\log 2}$

(3) for $0 \leqq \lambda \leqq 1$

$$\lim_{N\to\infty} \frac{1}{N} \#\{n;\ q_n|\alpha q_n - p_n| < \lambda,\ n \leqq N\} = \begin{cases} \frac{\lambda}{\log 2} & 0 \leqq \lambda \leqq \frac{1}{2} \\ \frac{-\lambda + \log 2\lambda + 1}{\log 2} & \frac{1}{2} \leqq \lambda \leqq 1 \end{cases}$$

(see [1])

(4) for $0 \leqq \lambda \leqq \frac{1}{2}$

$$\lim_{N\to\infty} \frac{1}{\log N} \#\{(q,p);\ q|q\alpha - p| < \lambda,\ (q,p)=1,\ q \leqq N\} = \frac{\pi^2}{12} \cdot \lambda$$

(see [5] and c.f. [2]

, where '=' means a.e.equality.

The main tool to obtain these theorems is the map $\bar{S}$ which will be called a natural extension of the algorithm $([0,1), S)$, and the ergodicity of the map $\bar{S}$.

Let $\bar{X}=[0,1)\times[0,1)$ and define a map $\bar{S}$ of $\bar{X}$ onto itself by

$$\bar{S}(\alpha,\beta) = (S\alpha, \frac{1}{a_1+\beta}) \quad \text{for } (\alpha,\beta)\in\bar{X},$$

then

Theorem 1 The map $\bar{S}$ is a one to one, onto map, and has an invariant measure $\bar{\mu}$ satisfying

$$d\bar{\mu} = \frac{1}{\log 2}\frac{d\alpha d\beta}{(1+\alpha\beta)^2} .$$

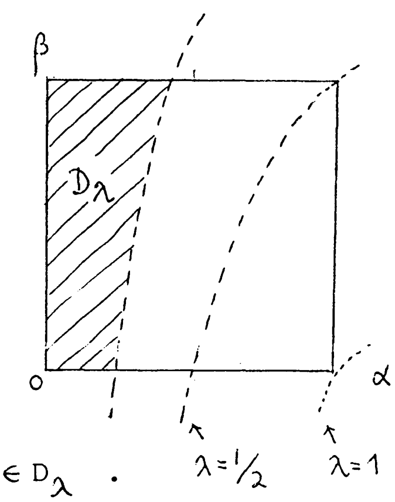
Moreover, the dynamical system $(\bar{X},\bar{S},\bar{\mu})$ is ergodic.

Using the natural extension, we obtain the proofs of metrical theorems. In fact, in the case of (3), from the definition of $\bar{S}$ and (1.4), we have

$$\bar{S}^n(\alpha,0) = (S^n\alpha, \frac{q_{n-1}}{q_n})$$

and

$$q_n|\alpha\cdot q_n-p_n| = \frac{S^n\alpha}{1+\frac{q_{n-1}}{q_n}\cdot S^n\alpha} .$$



Let $D_\lambda = \{(\alpha,\beta)\in\bar{X} \mid \frac{\alpha}{1+\alpha\cdot\beta} < \lambda\}$,

then

$$q_n|\alpha\cdot q_n - p_n| < \lambda \quad \text{iff} \quad \bar{S}^n(\alpha,0)\in D_\lambda .$$

Therefore, from the individual ergodic theorem,

$$\frac{1}{N}\#\{n;\ q_n|\alpha\cdot q_n-p_n|<\lambda,\ n\leqq N\}$$
$$=\frac{1}{N}\#\{n;\ \bar{S}^n(\alpha,0)\in D_\lambda,\ n\leqq N\} \xrightarrow[N\to\infty]{} \bar{\mu}(D_\lambda) \quad \text{a.e. } \alpha$$

Putting in order the procedure of the proof, we obtain the following scheme.

Procedure

approximation of α by $\frac{p_n}{q_n}$

Diophantine approximation

↓

$$S : \alpha \longrightarrow \frac{1}{\alpha} - \left[-\frac{1}{\alpha}\right]$$

Construction of the algorithm which induces the approximation

↓

$$\bar{X} := [0,1) \times [0,1)$$

$$\bar{S}(\alpha,\beta) := \left(S\alpha, \frac{1}{a_1+\beta}\right)$$

$$d\bar{\mu} := \frac{d\alpha\, d\beta}{\log 2\,(1+\alpha\beta)^2}$$

$(\bar{X}, \bar{S}, \bar{\mu})$ is ergodic

Construction of the Natural extension of the algorithm.

(Construction of the dynamical system)

↓

$$q_n|\alpha \cdot q_n - p_n| = \frac{S^n\alpha}{1+\frac{q_{n-1}}{q_n}\cdot S^n\alpha}$$

$$\bar{S}^n(\alpha, 0) = \left(S\alpha, \frac{q_{n-1}}{q_n}\right)$$

$$D_\lambda := \left\{(\alpha,\beta) \in \bar{X} : \frac{\alpha}{1+\alpha\beta} < \lambda\right\}$$

$$q_n|\alpha \cdot q_n - p_n| < \lambda \text{ iff } \bar{S}^n(\alpha, 0) \in D_\lambda$$

Interpretation of the terms of diophantine approx. as of ergodic theorey.

↓ application of ergodic theory

§ 2 The presentation of problems

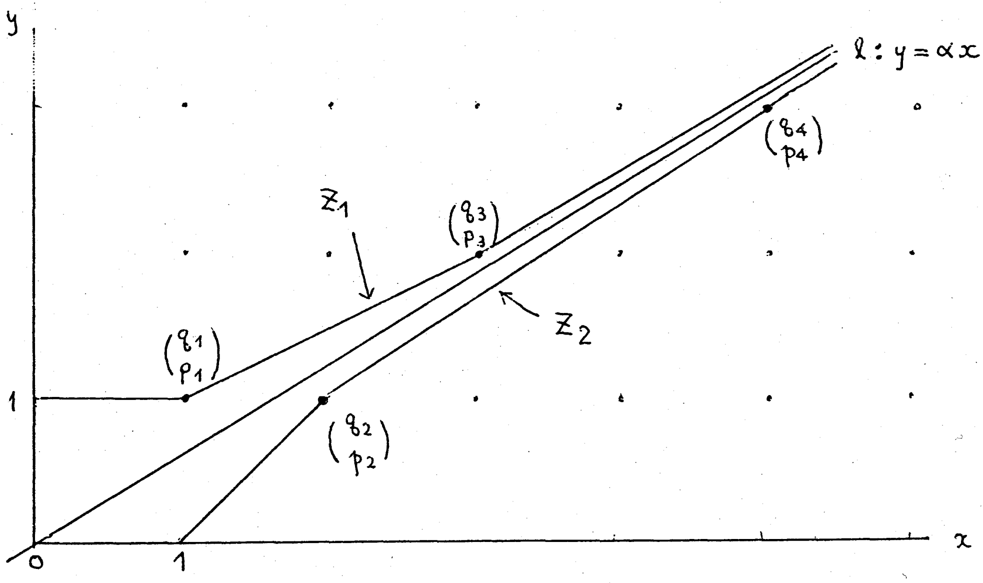
As a characterization of the principal convergents (q_n, p_n) $n=1,2,\cdots$, we know the convergents give a best approximation to α:

for any $A>0$,

$$\min_{0<q\leqq A} |q\cdot\alpha - p| = |q_n\cdot\alpha - p_n|$$

where n is chosen as $q_n \leqq A < q_{n+1}$.

Form the geometrical point of view, the convergents are also characterized as the vertices of approximating polygon Z_1 and Z_2 to the line l : $y=\alpha x$ (see figure).



The first presentation of problem is as follows.

Problem 1 : to construct the algorithm T_1 which induces a best approximation from below to α , that is, to construct the algorithm T_1 which induces the sequence (q_n', p_n') $n=1,2,\cdots$:

for any $A>0$,

$$\min_{\substack{0<q\leqq A \\ q\alpha-p>0}} (q\alpha - p) = q_n'\cdot\alpha - p_n'$$

where n is chosen as $q_n' \leqq A < q_{n+1}'$.

<u>Problem 2</u> : to construct the algorithm T_2 which induces <u>the mediant convergents</u>, that is, to construct the algorithm T_2 which induces the sequence (v_n , w_n) n=1,2,··· such that

$$\left\{ \frac{w_n}{v_n} : n = 1,2,\cdots \right\} = \bigcup_{m=1}^{\infty} \left\{ \frac{\lambda p_m + p_{m-1}}{\lambda q_m + q_{m-1}} \quad ; \quad \lambda = 1,2,\cdots, a_m \right\} .$$

The third problem is as follows .

<u>Problem 3</u> : to construct the algorithm T_3 which induces a best approximation for <u>inhomogeneous</u> linear form :

$$\min_{0<|x|<A} | \alpha x + \beta - y | = | x_n \cdot \alpha + \beta - y_n | \quad .$$

And then we should answer what theorem can be obtained (in metrical point of view) by using the algorithm?

§ 3 Result

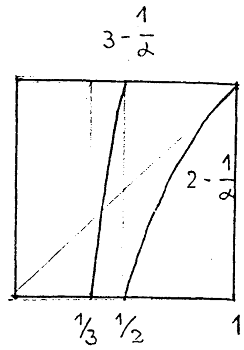
For problem 1 : The algorithm which induces the convergents from below to α : $\min\limits_{\substack{0<q<A \\ q\alpha - p>0}} (q\alpha - p) = q_n' \alpha - p_n'$

Algorithm : $X := [0,1)$

$$T_1\alpha := -\left[-\frac{1}{\alpha}\right] - \frac{1}{\alpha}$$

$$a_n(\alpha) := a(T_1^{n-1}\alpha) \quad \text{where } a(\alpha) := -\left[-\frac{1}{\alpha}\right].$$

$$(a_n \in \{2,3,4,\cdots\})$$



graph of T_1

Expansion and approximation :

$$\alpha = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \ddots \cfrac{1}{a_n - T_1^n\alpha}}} = \frac{p_n - T_1^n\alpha \cdot p_{n-1}}{q_n - T_1^n\alpha \cdot q_{n-1}}$$

where $\begin{pmatrix} q_n' & -q_{n-1}' \\ p_n' & -p_{n-1}' \end{pmatrix} := \begin{pmatrix} q_{n-1}' & -q_{n-2}' \\ p_{n-1}' & -p_{n-2}' \end{pmatrix} \begin{pmatrix} a_n & -1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} q_0' & -q_{-1}' \\ p_0' & -p_{-1}' \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

, then the sequence (q_n', p_n') $n=1,2,\cdots$ satisfies :

$$\min_{\substack{0<q<A \\ q\alpha - p>0}} (q\cdot\alpha - p) = q_n'\cdot\alpha - p_n'$$

where n is chosen as $q_n' \leqq A < q_{n+1}'$.

Natural extension : $\overline{X} := [0,1)\times[0,1)$

$$\overline{T}_1(\alpha,\beta) := \left(T_1\alpha , \frac{1}{a_1 - \beta}\right)$$

$$d\overline{\mu}_1(\alpha,\beta) := \frac{d\alpha\, d\beta}{(1-\alpha\beta)^2},$$

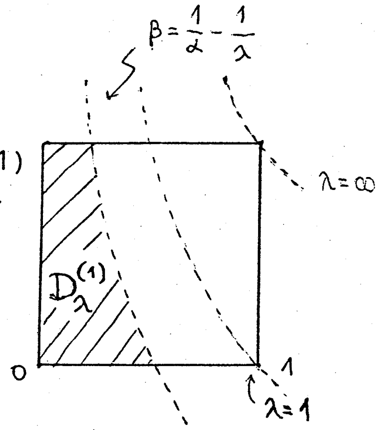
then $(\overline{X} , \overline{T}_1 , d\overline{\mu}_1)$ is ergodic .

<u>:nterpretation</u> : $q_n'(q_n'\alpha - p_n') = \dfrac{T_1^n\alpha}{1-T_1^n\alpha\cdot\dfrac{q_{n-1}'}{q_n'}}$

$$\overline{T}_1^n(\alpha,0) = \left(T_1^n\alpha,\ \frac{q_{n-1}'}{q_n'}\right),$$

therefore

$$q_n'(q_n'\alpha - p_n') < \lambda \quad \text{iff} \quad \overline{T}_1^n(\alpha,0) \in D_\lambda^{(1)}$$



where

$$D_\lambda^{(1)} = \left\{(\alpha,\beta)\in\overline{X} : \frac{\alpha}{1-\alpha\beta} < \lambda\right\}$$

<u>letrical theorem</u> : by ratio-ergodic theorem,

(1) $$\frac{\#\{n : q_n'(q_n'\alpha - p_n') < \lambda_1,\ n \leqq N\}}{\#\{n : q_n'(q_n'\alpha - p_n') < \lambda_2,\ n \leqq N\}} \xrightarrow[(N\to\infty)]{} \frac{\overline{\mu}_1(D_{\lambda_1}^{(1)})}{\overline{\mu}_1(D_{\lambda_2}^{(1)})}$$

where $\overline{\mu}_1(D_\lambda^{(1)}) = \begin{cases}\lambda & \text{if } 0<\lambda\leqq 1\\ 1+\log\lambda & \text{if } 1<\lambda\end{cases}$

(2) in paticular, if $\lambda_1, \lambda_2 \leqq 1$,

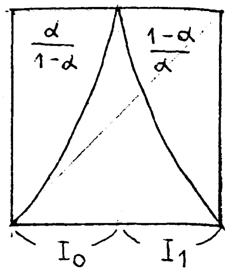
$$\frac{\#\{(q,p) : 0<q(q\cdot\alpha - p)<\lambda_1,\ q \leqq N\}}{\#\{(q,p) ; 0<q(q\cdot\alpha - p)<\lambda_2,\ q \leqq N\}} \xrightarrow[(N\to\infty)]{} \frac{\lambda_1}{\lambda_2}$$

For problem 2 : The algorithm which induces the mediant convergents.

Algorithm : $X := [0,1)$

$$T_2\alpha = \begin{cases} \dfrac{\alpha}{1-\alpha} & \text{if } \alpha \in I_0 = [0, \tfrac{1}{2}) \\ \dfrac{1-\alpha}{\alpha} & \text{if } \alpha \in I_1 = [\tfrac{1}{2}, 1) \end{cases}$$

$$\varepsilon_n(\alpha) := \varepsilon \quad \text{if } T_2^{n-1}\alpha \in I_\varepsilon .$$



Expansion and approximation :

$$\begin{pmatrix} r_n & s_n \\ t_n & u_n \end{pmatrix} := A_{\varepsilon_1(\alpha)} A_{\varepsilon_2(\alpha)} \cdots A_{\varepsilon_n(\alpha)}$$

where $A_0 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A_1 := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

, then

$$\alpha = \frac{t_n + T_2^n\alpha \cdot u_n}{r_n + T_2^n\alpha \cdot s_n} \qquad \text{(expansion)}$$

$$\left\{ \frac{w_n}{v_n} : n=1,2,\cdots \right\} \qquad \text{(sequence of mediant conv.)}$$

where $w_n := t_n + u_n \qquad v_n := r_n + s_n$

Natural extension : $\overline{X} := [0,1)\times[0,1)$

$$\overline{T}_2(\alpha,\beta) := \begin{cases} \left(\dfrac{\alpha}{1-\alpha}, \dfrac{\beta}{1+\beta}\right) & \text{if } \alpha \in I_0 \\ \left(\dfrac{1-\alpha}{\alpha}, \dfrac{1}{1+\beta}\right) & \text{if } \alpha \in I_1 \end{cases}$$

$$d\overline{\mu}_2 = \frac{d\alpha\, d\beta}{(\alpha + \beta - \alpha\beta)^2}$$

then $(\overline{X}, \overline{T}_2, d\overline{\mu}_2)$ is ergodic.

Interpretation :

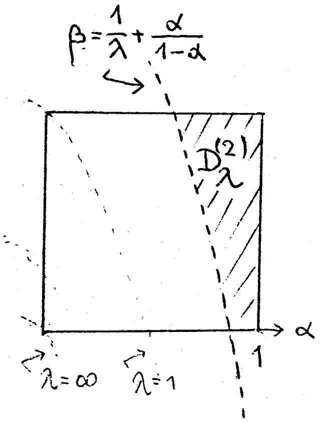
$$v_n | v_n \alpha - w_n | = \frac{1 - T_2^n \alpha}{\frac{r_n}{v_n}(1 - T_2^n \alpha) + T_2^n \alpha}$$

$$\overline{T}_2^n(\alpha, 1) = (T_2^n \alpha, \frac{r_n}{v_n})$$

therefore,

$$v_n | v_n \alpha - w_n | < \lambda \text{ iff } \overline{T}_2^n(\alpha, 1) \in D_\lambda^{(2)}$$

where $D_\lambda^{(2)} = \{ (\alpha, \beta) \in \overline{X} ; \frac{1-\alpha}{\beta(1-\alpha)+\alpha} < \lambda \}$



Metrical theorem [4] : for a.e. ,

$$(1) \quad \frac{\#\{n ; v_n | v_n \alpha - w_n | < \lambda_1, n \leqq N\}}{\#\{n ; v_n | v_n \alpha - w_n | < \lambda_2, n \leqq N\}} \longrightarrow \frac{\overline{\mu}_2(D_{\lambda_1}^{(2)})}{\overline{\mu}_2(D_{\lambda_2}^{(2)})}$$

where

$$\overline{\mu}_2(D_\lambda^{(2)}) = \begin{cases} \lambda & \text{if } \lambda \leqq 1 \\ \log\lambda + 1 & \text{if } \lambda \geqq 1 \end{cases} .$$

In paticular, if $\lambda_1, \lambda_2 \leqq 1$, then

$$(2) \quad \frac{\#\{(q,p) ; |q(q\alpha - p)| < \lambda_1 , q \leqq N\}}{\#\{(q,p) ; |q(q\alpha - p)| < \lambda_2 , q \leqq N\}} \longrightarrow \frac{\lambda_1}{\lambda_2} \quad \text{(a.e.)} .$$

By inducing modified algorithm of T_2, we have

(3) for $\lambda \leqq 1$,

$$\frac{\#\{(q,p) ; |q(q\alpha - p)| < \lambda, q \leqq N\}}{\log N} \longrightarrow \frac{12}{\pi^2} \cdot \lambda \quad \text{(a.e.)} .$$

For problem 3 : the algorithm which induces the approximation for in-homogeneous linear class :

$$\min_{\substack{(x,y)\\ 0<|x|<A}} |x(\alpha x+\beta-y)| = |\alpha x_n+\beta-y_n| .$$

Algorithm : $X := \{(\alpha,\beta) : 0 \leqq \beta < 1 ,\quad -\beta \leqq \alpha < -\beta+1\}$

$$T_3(\alpha,\beta) := \left(\frac{1}{\alpha} - a(\alpha,\beta) ,\ b(\alpha,\beta) - \frac{\beta}{\alpha}\right)$$

where $a(\alpha,\beta) := \left[\frac{1-\beta}{\alpha}\right] - \left[-\frac{\beta}{\alpha}\right]$ $\qquad b(\alpha,\beta) := -\left[-\frac{\beta}{\alpha}\right]$

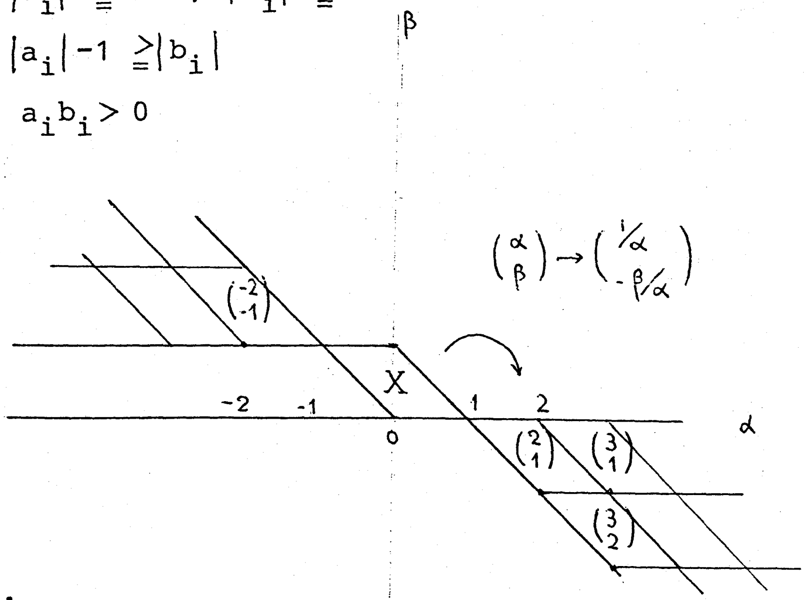
digits $\begin{pmatrix} a_k \\ b_k \end{pmatrix} := \begin{pmatrix} a(T_3^{k-1}(\alpha,\beta)) \\ b(T_3^{k-1}(\alpha,\beta)) \end{pmatrix}$

where digits a_k and b_k satisfies

(1) $|a_i| \geqq 2$, $|b_i| \geqq 1$

(2) $|a_i|-1 \geqq |b_i|$

and (3) $a_i b_i > 0$



Approximation :

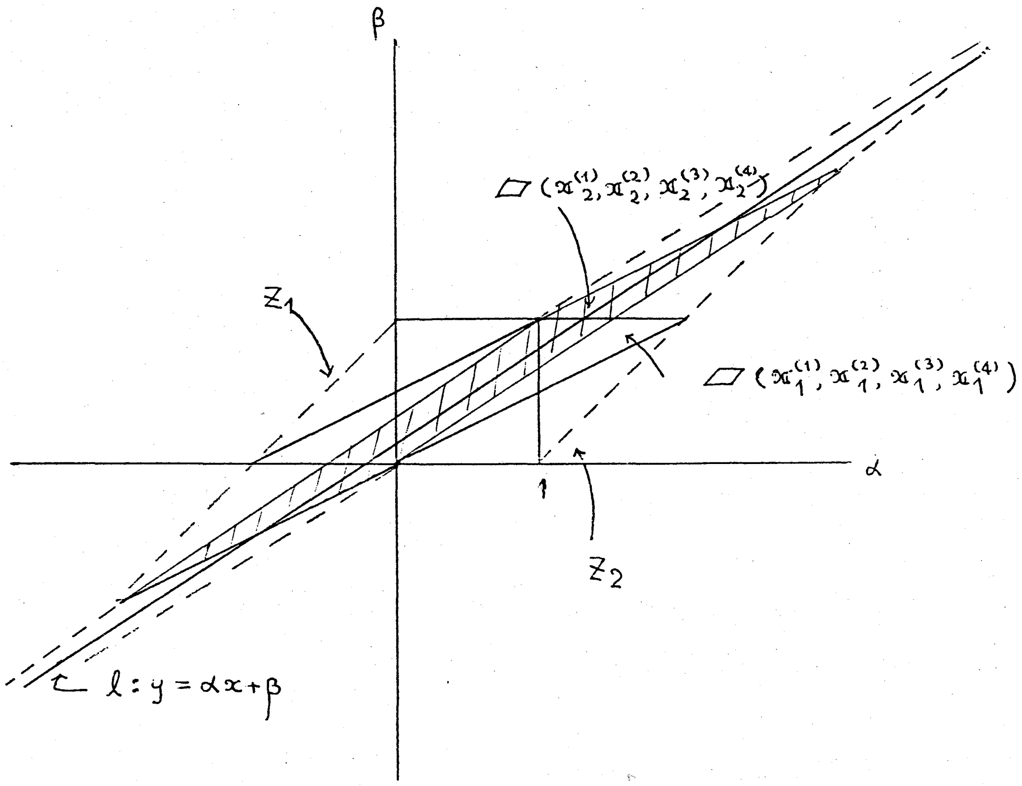
$$\begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} := \begin{pmatrix} q_{n-1} & q_{n-2} \\ p_{n-1} & p_{n-2} \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

$$\varphi_n\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \sum_{k=0}^{n-1} q_k b_{k+1} \\ \sum_{k=0}^{n-1} p_k b_{k+1} \end{pmatrix} ,$$

then points

$$\mathbf{x}_n^{(i)} := \begin{pmatrix} x_n^{(i)} \\ y_n^{(i)} \end{pmatrix} := \varphi_n(\mathbf{a}_i) \qquad i=1,2,3,4$$

, where $\mathbf{a}_1=\begin{pmatrix}0\\0\end{pmatrix}$, $\mathbf{a}_2=\begin{pmatrix}1\\0\end{pmatrix}$, $\mathbf{a}_3=\begin{pmatrix}1\\1\end{pmatrix}$, $\mathbf{a}_4=\begin{pmatrix}0\\1\end{pmatrix}$, give an approximating parallelogram, and $\mathbf{x}_n^{(i)}$: $i=1,2,3,4$ $n \in N$ gives the approximating polygons Z_1 and Z_2 to the line $l : y=\alpha x+\beta$. (see figure)



<u>Natural extension</u> : $\overline{X} := X \times X^*$

where $X^* := \{ (\gamma, \delta) : 0 \leq \delta < 1 \text{ and } 0 < \delta - \gamma < 1 \}$

$$\overline{T}_3(\alpha, \beta, \gamma, \delta) := \left(T_3(\alpha, \beta), \frac{1}{a_1+\gamma}, \frac{b_1+\delta}{a_1+\gamma} \right)$$

$$d\overline{\mu}_3 := \frac{1}{\log 2} \frac{d\alpha\, d\beta\, d\gamma\, d\delta}{(1+\alpha\gamma)^3},$$

$(\overline{X}, \overline{T}_3, \overline{\mu}_3)$ is ergodic.

Interpretation : Put $\bar{T}_3^n(\alpha, \beta, 0, 0) = (\alpha_n, \beta_n, \gamma_n, \delta_n)$, then

$$|x_n^{(i)}(\alpha x_n^{(i)} + \beta - y_n^{(i)})| = \begin{cases} \frac{1}{1+\alpha_n\gamma_n}\delta_n\beta_n & \text{if } i=1 \\ \frac{1}{1+\alpha_n\gamma_n}(\delta_n-\gamma_n)(1-\beta_n) & \text{if } i=2 \\ \frac{1}{1+\alpha_n\gamma_n}(1+\gamma_n-\delta_n)(1+(\alpha_n+\beta_n)) & \text{if } i=3 \\ \frac{1}{1+\alpha_n\gamma_n}(1-\delta_n)(\alpha_n+\beta_n) & \text{if } i=4 \end{cases},$$

therefore,

$$\min_{i=1,2,3,4} |x_n^{(i)}(\alpha x_n^{(i)} + \beta - y_n^{(i)})| < \lambda \text{ iff } \bar{T}_3^n(\alpha, \beta, 0, 0) \in \bigcup_{i=1,2,3,4} D_i(\lambda)$$

where

$$D_1(\lambda) = \{(\alpha, \beta, \gamma, \delta) \in \bar{X} : \frac{\delta\beta}{1+\alpha\gamma} < \lambda\}$$

$$D_2(\lambda) = \{(\alpha, \beta, \gamma, \delta) \in \bar{X} : \frac{(\delta-\gamma)(1-\beta)}{1+\alpha\gamma} < \lambda\}$$

$$D_3(\lambda) = \{(\alpha, \beta, \gamma, \delta) \in \bar{X} : \frac{(1+\gamma-\delta)(1+\alpha+\beta)}{1+\alpha\gamma} < \lambda\}$$

$$D_4(\lambda) = \{(\alpha, \beta, \gamma, \delta) \in \bar{X} : \frac{(1-\delta)(\alpha+\beta)}{1+\alpha\gamma} < \lambda\}$$

Metrical theorem [3] : for a.e. (α, β),

(1) $$\lim_{n\to\infty} -\frac{1}{n}\log \min_{i=1,2,3,4}\{|\alpha x_n^{(i)} + \beta - y_n^{(i)}|\} = \frac{\pi^2}{12\log 2}$$

(2) $$\lim_{n\to\infty}\frac{1}{N}\#\{n \,;\, \min_{i=1,2,3,4}|x_n^{(i)}(\alpha x_n^{(i)} + \beta - y_n^{(i)})| < \lambda,\ n \leq N\} = G(\lambda)$$

where $$G(\lambda) = \frac{1}{\log 2}\int_{\bigcup_{i=1,2,3,4} D_i(\lambda)} \frac{d\alpha\, d\beta\, d\gamma\, d\delta}{(1+\alpha\gamma)^3}$$

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