数理解析研究所講究録
第 635 巻 1987 年 145-164

Quasiconformal surgery on doubly attractive cycles

Shigehiro Ushiki 宇敷重広
Institute of Mathematics, Yoshida College
Kyoto University, Kyoto, 606 Japan

Abstract

A quasiconformal surgery is executed on a rational function with an attractive cycle which attracts two critical points. As the result of the surgery, a rational function with a doubly attractive cycle of different period is obtained. The quasiconformal suegery reveals that the topological structure of the Julia set of the obtained function has a self-similar lattice structure.

§0. INTRODUCTION

As the understanding of the dynamics on the complex plane of the family of quadratic functions gave a deep insight into the iteration of unimodal maps on the interval, complexified maps may help us to understand the real analytic mappings of the circle.

In section 1, we study the case where the complex rational function has a (doubly) super-attractive cycle of period two.

In section 2, we treat the case when a fixed point is attractive. A quasiconformal surgery is used to understand the Julia set of the system.

In section 3, a quasiconformal surgery is executed on a function with a (doubly) super-attractive cycle of period two, to construct a new function with a (doubly) super-attractive cycle of period three. This surgery gives a combinatorial description of the Julia set of the obtained rational function.

The combinatorial structure as a "self-similar lattice" of the Julia set is described in section 4.

This note is an abreviated version of [21].

§1. BLASCHKE'S FUNCTION OF DEGREE TWO

Let $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ denote the Riemann sphere and let $\mathbf{D} = \{ z \in \mathbf{C} \mid |z| < 1 \}$ denote the interior of the unit disk. The unit circle will be denoted by $\partial\mathbf{D}$. Let $f(z)$ be a rational function with complex coefficients. If $f(z) = P_1(z)/P_2(z)$, where $P_1(z)$ and $P_2(z)$ are polynomials without common factor, then

$$\deg(f) = \sup(\deg(P_1(z)),\deg(P_2(z)))$$

is called the degree of f.

Rational function $f(z)$ defines a dynamical system $f : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ on the Riemann sphere. For an integer k, $f^k = f\circ f\circ\cdots\circ f$ denotes the k-times iterated composition of f.

A point $c \in \bar{\mathbf{C}}$ is called a *critical point* if $f'(c) = 0$ (in an appropriate coordinate). A point $p \in \bar{\mathbf{C}}$ is called a *periodic point* if $f^k(p) = p$ for some positive integer k. The smallest positive integer k with $f^k(p) = p$ is called the period of p. The orbit $\{p, f(p),\dots,f^{k-1}(p)\}$ of the periodic point is called a *cycle*. Periodic point of period one is called a *fixed point*.

If $P = \{p,f(p),\dots,f^{k-1}(p)\}$ is a cycle of period k, the value

$$\sigma(P) = \prod_{j=0}^{k-1} f'(f^j(p))$$

is called the *multiplicator* of the cycle. The multiplicator is defined by an appropriate choice of coordinate of the Riemann sphere and does not depend on the choice.

If $|\sigma(P)| > 1$ then the cycle P is said *repulsive*. If $|\sigma(P)| = 1$ then P is said *neutral*. If $|\sigma(P)| < 1$ then P is said *attractive*. If P is an attractive cycle of period k and $p \in P$, then the *attrac-*

tive basin A(p) is defined as

$$A(p) = \{z \in \bar{\mathbf{C}} \mid f^{nk}(z) \to p \text{ as } n\to\infty\}.$$

The *immediate attractive basin* $A^*(p)$ is the connected component of A(p) containing p. The attractive basin and the immediate basin of an attractive cycle P are defined respectively by

$$A(P) = \bigcup_{j=0}^{k-1} A(f^j(p)) \quad \text{and} \quad A^*(P) = \bigcup_{j=0}^{k-1} A^*(f^j(p)).$$

A cycle, P, is said to be *super-attractive* if its multiplicator vanishes, *i.e.*, if P contains a critical point. If a cycle, P, contains two distinct critical points, P is said to be *doubly super-attractive*. If P attracts two critical points, P is said to be *doubly attractive*.

If $c \in \bar{\mathbf{C}}$ is a critical point and its orbit $O(c) = \{c, f(c), f^2(c),...\}$ contains another critical point, then c is said to be *doubly critical*.

Let $f : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ be a degree-two complex rational function of Blaschke's type

$$f(z) = z\cdot\frac{z+\lambda}{1+\bar{\lambda}z}, \tag{1}$$

where $\lambda \in \mathbf{C}$ is a parameter. (1) can be considered as a real two-parameter family of dynamical systems on the Riemann sphere. Note that f maps the unit circle $\partial\mathbf{D}$ into itself. If $|\lambda| < 1$ then $f(\mathbf{D}) = \mathbf{D}$ and $f(\bar{\mathbf{C}}-\bar{\mathbf{D}}) = \bar{\mathbf{C}}-\bar{\mathbf{D}}$, and f maps the unit circle $\partial\mathbf{D}$ onto itself with topological degree two. In this case, f has two attractive fixed points, 0 and ∞, with

$$A^*(0) = \mathbf{D} \text{ and } A^*(\infty) = \bar{\mathbf{C}}-\bar{\mathbf{D}}.$$

If $|\lambda| = 1$, then (1) reduces to a linear rotation $f(z) = \lambda z$.

Now, let us consider the case $|\lambda| > 1$. In this case, the topological mapping degree of f restricted to the unit circle is zero. The mapping f has three fixed points, 0, ∞, and $\alpha = (\lambda-1)/(\bar{\lambda}-1)$. Note that $\alpha \in \partial\mathbf{D}$. The differential of f is given by

$$f'(z) = \frac{\bar{\lambda}z^2+2z+\lambda}{(1+\bar{\lambda}z)^2}. \quad (2)$$

The multiplicators of these fixed points are, respectively, λ, $\bar{\lambda}$, and $(\lambda+\bar{\lambda}-2)/(\lambda\bar{\lambda}-1)$.

The mapping f is "mirror" symmetric with respect to the unit circle in the sense $f(z) = \phi\circ f\circ\phi(z)$, where $\phi(z) = 1/\bar{z}$. If $|\lambda+1| > 2$ then fixed point α is attractive. In this case, the attractive fixed point attracts both of the two critical points :

$$c_\gamma = \frac{-1+\gamma\sqrt{\lambda\bar{\lambda}-1}\,i}{\bar{\lambda}}, \quad \gamma = \pm 1, \quad (3)$$

and the Julia set J_f is a Cantor set. This fact can be verified by using the quasiconformal surgery explained later. The critical points c_γ will be denoted as c_+ and c_- for $\gamma = +1$ and -1 respectively.

Next, let us consider the periodic point of period two. Periodic points of period two are given as solutions of equation

$$f^2(z) - z = 0. \quad (4)$$

Since fixed points satisfy this equation, too, we have only one cycle of period two. By noting that (4) can be factorized by $f(z) - z$, we get the following quadratic equation for the 2-periodic points :

$$(\bar{\lambda}+1)z^2 + (\lambda+1)(\bar{\lambda}+1)z + \lambda+1 = 0. \quad (5)$$

This equation has multiple root $z = -(\lambda+1)/2$ if λ lies on the circle $|\lambda+1| = 2$, where period doubling bifurcation occurs.

If $0 < |\lambda+1| < 2$, then (5) has two distinct roots $b_\nu \in \partial\mathbf{D}$, $\nu = \pm 1$, which are given by

$$b_\nu = \frac{-|\lambda+1|^2+\nu\sqrt{4|\lambda+1|^2-|\lambda+1|^4}\,i}{2(\bar{\lambda}+1)}. \quad (6)$$

We see immediately that $b_\nu = f(b_{-\nu})$. The multiplicator, $\sigma_2(\lambda) = \sigma(\{b_\nu\}_{\nu=\pm 1})$ of the 2-cycle is computed as

$$\sigma_2(\lambda) = f'(b_+)\cdot f'(b_-) = \frac{|\lambda|^2-5+(|\lambda+1|^2-2)^2}{|\lambda|^2-1},$$

where $b_\pm = b_{\pm 1}$.

Proposition 1.1. The set of parameters $\{\lambda \in \mathbf{C} \mid -1 < \sigma_2(\lambda) < 1\}$, where f has an attractive 2-cycle, is a simply connected region. Its boundary consists of real algebraic curves :

$$|\lambda+1| = 2 \quad \text{for} \quad \sigma_2(\lambda) = 1$$

and

$$2|\lambda|^2 - 6 + (|\lambda+1|^2-2)^2 = 0 \quad \text{for} \quad \sigma_2(\lambda) = -1.$$

By setting $\lambda = \xi+i\eta$ and $R = \lambda\bar{\lambda}$, these curves can be rewritten as

$$R = 3 - 2\xi \quad \text{for} \quad \sigma_2(\lambda) = 1$$

and

$$4\xi + 5 = (R+2\xi)^2 \quad \text{for} \quad \sigma_2(\lambda) = -1.$$

In general, for $\sigma \in (-1,1)$, the set $\{\lambda \in \mathbf{C} \mid \sigma_2(\lambda) = \sigma\}$ is given by a parabola in the (R,ξ) coordinate :

$$(R+2\xi)^2 = 4\xi + (1+\sigma)R + 4 - \sigma.$$

We denote this "mushroom" region of the proposition above by $W_{1/2}$.

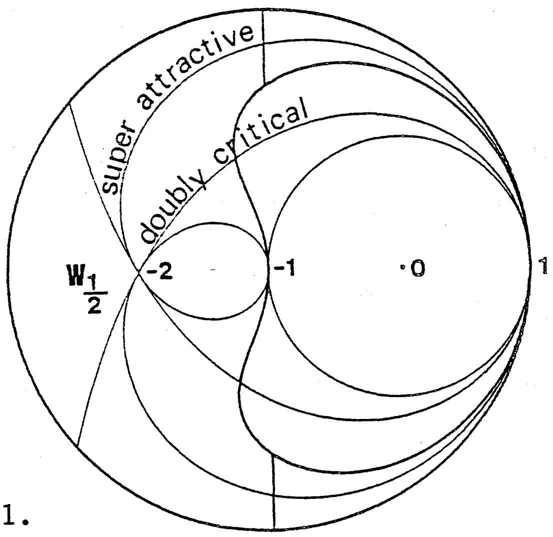


Fig.1.1.

The values of parameter λ, for which the 2-cycle is super-attractive, are given by the equation

$$\sigma_2(\lambda) = 0, \ \lambda \in W_{1/2} \text{ (see Fig.1.1).}$$

Let $\lambda = \xi + \eta i$, $\xi, \eta \in \mathbf{R}$, and let $R = \lambda\bar{\lambda}$. Following propositions can be verified by direct computations.

Proposition 1.2. If $\lambda \in w_{1/2,0/1}$, then $b_+ = c_+$. If $\lambda \in w_{1/2,1/1}$, then $b_- = c_-$. (See fig.1.2.)

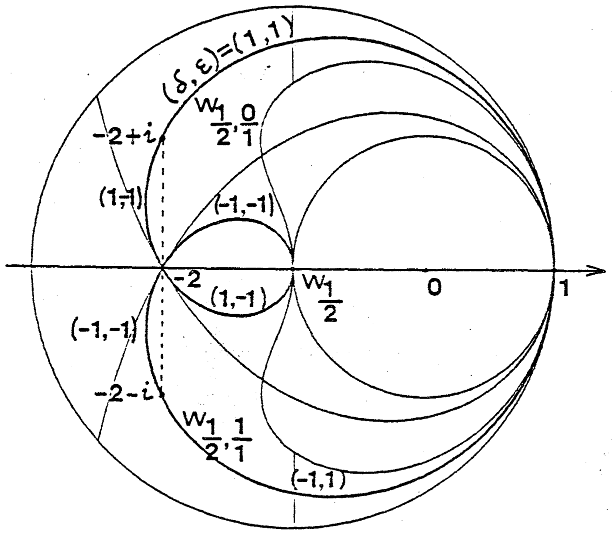


Fig.1.2.

Proposition 1.3. The locus of parameter λ, for which $f(c_\nu) = c_{-\nu}$ holds, is given by

$$\lambda = (1+\nu\sqrt{R-1}\,i)^3/R, \quad R > 1.$$

Proposition 1.4. The locus of parameter λ, for which $f^2(c_\nu) = \alpha$ and $\alpha \neq c_\nu$ hold, is given by

$$\xi = -1, \quad \eta = -\nu\sqrt{R-1}, \quad R > 1.$$

Let $V_{1/2} = \{\lambda \in \mathbf{C} \mid \lambda \in W_{1/2}, \ \mathcal{R}e\lambda < -1\}$.

Theorem 1.5. The 2-cycle of f attracts both of the two critical points if and only if $\lambda \varepsilon V_{1/2}$. Moreover, if $\lambda \varepsilon V_{1/2}$, then $c_\nu \varepsilon A^*(b_\nu)$, $\nu = \pm 1$, and the Julia set is a Jordan curve.

See [21] for the proof.

Suppose $\lambda \varepsilon V_{1/2}$ and $\phi_\nu : \mathbf{D}_\nu \to \mathbf{D}_{-\nu}$ be the Blaschke product in the proof above (case i)). Let $\kappa_\nu \varepsilon \mathbf{D}_\nu$ be the critical point of ϕ_ν :

$$\kappa_\nu = \frac{-1+\sqrt{1-|m_\nu|^2}}{\bar{m}_\nu}.$$

As we consider the Blaschke's family (1), we see that $-1 < \kappa_\nu < 1$, $\nu = \pm 1$ for $\lambda \varepsilon V_{1/2}$. Hence $\kappa(\lambda) = (\kappa_+, \kappa_-)$ defines a mapping $\kappa : V_{1/2} \to I_+ \times I_-$, $I_\pm = (-1,1)$.

Theorem 1.6. The mapping κ is a real analytic diffeomorphism of $V_{1/2}$ onto $I_+ \times I_-$.

See [21] for the proof.

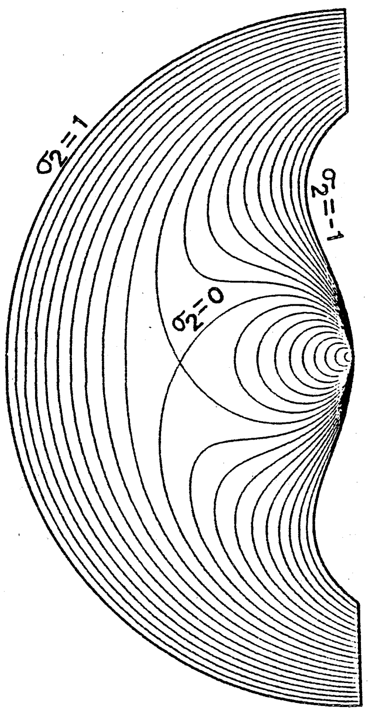


Fig.1.3.

Figure 1.3 shows the level curves of σ_2 in $V_{1/2}$. In $I_+ \times I_-$, the κ_ν axis, $I_+ \times \{0\}$ and $\{0\} \times I_-$ represent the parameters for which the attractive cycle is super-attractive. The curves of doubly critical cycles are given by $\kappa_\nu = -\kappa_{-\nu}^2$.

Corresponding level curves in $I_+ \times I_-$ are given in Fig.1.4. Curves for parameters with a doubly critical point are also shown.

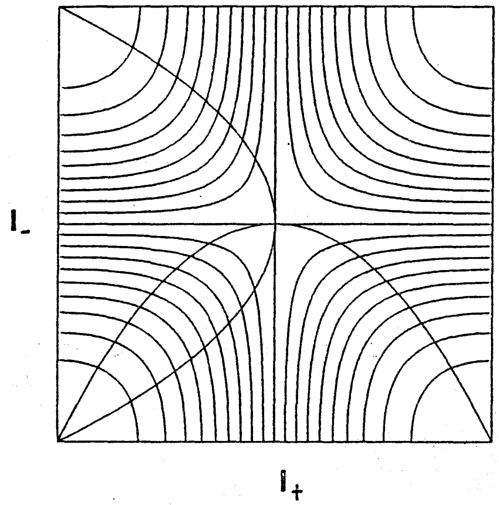


Fig.1.4.

§2. TOTALLY DISCONNECTED JULIA SET

If Blaschke's function (1) has an attractive fixed point and if both of the two critical points are contained in the immediate attractive basin of the attractive fixed point, then its Julia set is a totally disconnected Cantor set.

Let us recall Shishikura's fundamental lemma for quasiconformal mappings[17], which is an improved version of the straightening theorem of Douady and Hubbard[10].

Definition 2.1. Let Ω and Ω' be domains of **C**. A homeomorphism $\phi : \Omega \to \Omega'$ is a *quasiconformal mapping* (qc-mapping) if

1) ϕ is absolutely continuous on almost all lines parallel to the

real axis and almost all lines parallel to the imaginary axis;

2) for some constant $k < 1$,

$$\left|\frac{\partial\phi}{\partial \bar{z}} \Big/ \frac{\partial\phi}{\partial z}\right| \leq k$$

holds almost everywhere with respect to the Lebesgue measure.

Quasiconformal mappings on Riemann surfaces are defined by means of local coordinates.

Definition 2.2. A *quasi-regular* mapping is a composite of a quasi-conformal mapping and an analytic mapping.

Lemma 2.3.(Shishikura) Let $g : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ be a quasi-regular mapping. Let E_i, $i=1,\ldots,m$, be disjoint open subsets in $\bar{\mathbf{C}}$, and let $\phi_i : E_i \to E_i'$ be quasiconformal mappings, with E_i' open subsets of $\bar{\mathbf{C}}$. Let N be a non-negative integer. Assume the following conditions hold:

(i) $g(E)$ E, where $E = \bigcup_{i=1}^{m} E_i$;

(ii) $\Phi \circ g \circ \phi_i^{-1}$ is analytic on E_i', where $\Phi : E \to \bar{\mathbf{C}}$ is the union of the ϕ_i;

(iii) $\frac{\partial g}{\partial \bar{z}} = 0$ a.e. on $\bar{\mathbf{C}} - g^{-N}(E)$.

Then there exists a quasiconformal mapping $h : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ such that $h \circ g \circ h^{-1} : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ is a rational function. Moreover, $h \circ \phi_i^{-1}$ is conformal on E_i' and $\frac{\partial h}{\partial \bar{z}} = 0$ a.e. on $\bar{\mathbf{C}} - \bigcup_{n \geq 0} g^{-n}(E)$.

See [17] for the proof.

Proposition 2.4. If $|\lambda+1| > 2$ then both of the two critical points $c_{\pm}$ are contained in the immediate attractive basin $A^*(\alpha)$ of the fixed point $\alpha = (\lambda-1)/(\bar{\lambda}-1)$.

See [21] for the proof.

§3. SURGERY ON BLASCHKE'S FUNCTION WITH DOUBLY SUPER-ATTRACTIVE CYCLE

Let us begin by looking at the dynamics of $f_{-2} : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$, which has a doubly super-attractive cycle of period two. We see that $b_\nu = c_\nu = (1-\nu\sqrt{3}i)/2$, $\nu = \pm 1$. Möbius transformation $h(z) = (z-b_+)/(1-z/b_-)$ conjugates f_{-2} into $g(\zeta) = h \circ f_{-2} \circ h^{-1}(\zeta) = 1/\zeta^2$. The conjugacy map h maps the doubly super-attractive cycle $\{b_\pm\}$ to $\{0,\infty\}$. The Julia set of f_{-2} is $\mathbf{R} \cup \{\infty\}$, which is mapped onto the unit circle by the conjugacy map. The fixed points, 0, ∞, and $\alpha = 1$, of f_{-2} are mapped into $-b_+ = (-1+\sqrt{3}i)/2$, $-b_- = (-1-\sqrt{3}i)/2$, and 1 respectively by h. These points are fixed points of g. The unit circle is mapped onto $\mathbf{R} \cup \{\infty\}$.

Let $L = \mathbf{C}/\mathbf{Z}$ and define $G : L \to \bar{\mathbf{C}}$ by

$$G(\zeta) = h^{-1}(\exp(2\pi i\zeta)).$$

Then G gives an analytic conjugacy map between "linear" map $\tilde{f} : L \to L$, $\tilde{f}(\zeta) = -2\zeta$, and f_{-2}, i.e.,

$$f \circ G = G \circ \tilde{f}.$$

The mapping G omits only the super-attractive 2-cycle $\{b_\pm\}$ and it maps L isomorphically onto $\bar{\mathbf{C}} - \{b_\pm\}$.

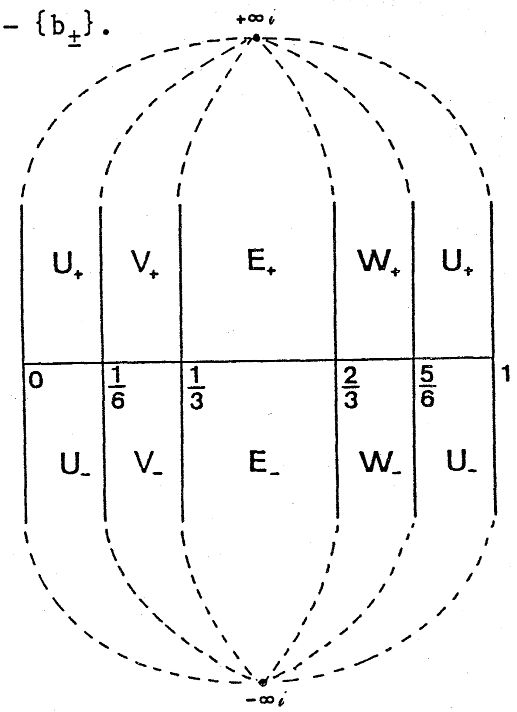


Fig. 3.1. Riemann surface $\bar{L}$. Vertical lines $\mathcal{Re}(\zeta) = 0$ and $\mathcal{Re}(\zeta) = 1$ are identified.

The multiplication map $\tilde{f}$ has three fixed points $\zeta = 1/3$, $2/3$, and 0, corresponding respectively to the fixed points of $\tilde{f}$, $0, \infty$, and $\alpha = 1$. The "real line" $\{\zeta \in L \mid \mathit{Im}(\zeta) = 0\}$ is the Julia set of $\tilde{f}$. It is mapped onto the "real line" $\mathbf{R} \cup \{\infty\}$ by G. The set $\{\zeta \in L \mid \mathit{Re}(\zeta) = 0$ or $1/2\}$ is invariant under $\tilde{f}$ and mapped into the unit circle by G. The "mirror symmetry" of f_{-2} with respect to the unit circle becomes a mirror symmetry $\zeta \quad -\bar{\zeta}$ on L. Let $\bar{L} = L \cup \{\pm\infty i\}$. Define closed regions E_ν, U_ν, V_ν, W_ν, $\nu = \pm 1$, in $\bar{L}$ as follows

$$E_\nu = \{\zeta \in L \mid 1/3 \leq \mathit{Re}\zeta \leq 2/3,\ \nu \mathit{Im}\zeta \geq 0\} \cup \{\nu\infty i\}$$
$$U_\nu = \{\zeta \in L \mid -1/6 \leq \mathit{Re}\zeta \leq 1/6,\ \nu \mathit{Im}\zeta \geq 0\} \cup \{\nu\infty i\}$$
$$V_\nu = \{\zeta \in L \mid 1/6 \leq \mathit{Re}\zeta \leq 1/3,\ \nu \mathit{Im}\zeta \geq 0\} \cup \{\nu\infty i\}$$
$$W_\nu = \{\zeta \in L \mid 1/3 \leq \mathit{Re}\zeta \leq 5/6,\ \nu \mathit{Im}\zeta \geq 0\} \cup \{\nu\infty i\}.$$

Observe that vertical lines $\mathit{Re}(\zeta) = 0$, $1/3$, $2/3$ are invariant under $\tilde{f}$. We see that each region is covered twice by $\tilde{f}$:

$$\tilde{f}(E_\nu) = \tilde{f}(U_\nu) = U_{-\nu} \cup V_{-\nu} \cup W_{-\nu},$$
$$\tilde{f}(V_\nu) = \tilde{f}(W_\nu) = E_{-\nu}.$$

If we parametrize vertical lines $\mathit{Re}(\zeta) =$ const. by $y = \mathit{Im}(\zeta)$, then $\tilde{f}$ induces a linear multiplication $y \mapsto -2y$ on these invariant vertical lines. Let $E = E_+ \cup E_-$, $U = U_+ \cup U_-$, $V = V_+ \cup V_-$, and $W = W_+ \cup W_-$.

Construct a Riemann surface X as follows. Make two copies $E^{(1)}$, $E^{(2)}$ of E and denote $z^{(1)} \in E^{(1)}$ and $z^{(2)} \in E^{(2)}$ for points corresponding to $z \in E$. The space X is obtained from the disjoint union

$$(\bar{L} - \mathrm{int}(E)) \amalg E^{(1)} \amalg E^{(2)}$$

by identifying:

$z \in \partial E \cap E_-$ with $z^{(1)} \in E^{(1)}$,

$z \in \partial E \cap E_+$ with $z^{(2)} \in E^{(2)}$,

and $z^{(2)} \in E^{(2)}$ with $\tilde{f}(z)^{(1)} \in E^{(1)}$ for $z \in \partial E \cap E_-$.

The natural confomal structure of $\bar{L}$ induces a conformal structure on X except at singular points $1/3$, $2/3$, and $p = +\infty i^{(1)}$ $(= -\infty i^{(2)})$. We can give an appropriate conformal structure at these points so that X is a Riemann shere.

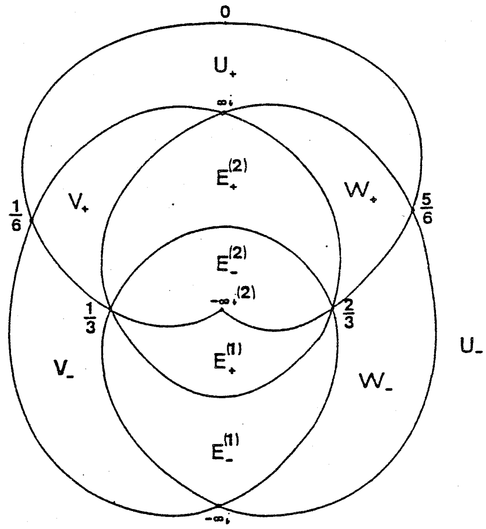


Fig.3.2. Riemann surface X.

Define a mapping $g_0 : X - \mathrm{int}(U_-) \to X$ by

$$g_0(z) = \tilde{f}(z) \quad \text{for } z \in U_+,$$

$$g_0(z) = \tilde{f}(z)^{(1)} \quad \text{for } z \in V \cup W,$$

$$g_0(z^{(1)}) = z^{(2)} \quad \text{for } z \in E,$$

$$g_0(z^{(2)}) = \tilde{f}(z) \quad \text{for } z \in E.$$

Note that this mapping is continuous and **C**-analytic on $X - U_-$. In order to extend the conformal mapping g_0 to a quasi-regular mapping $g : X \to X$, we need looking at the dynamics of $\tilde{f} : \bar{L} \to \bar{L}$ near the Julia set and in the attractive basin of the attractive cycle.

For complex numbers z_1, z_2, z_3, we denote by $\Delta(z_1,z_2,z_3)$ the closed triangle region in $\bar{L}$ obtained by projecting the triangle region whose vertices are these three points z_1, z_2, and z_3.

Choose a constant δ with $0 < \delta < 1/\sqrt{3}$. Let

$\Delta_1 = \Delta(-1/3, 1/3, -\delta i/3)$,

$\Delta_2 = \Delta(-1/3, 1/3, \delta i/6)$,

$\Delta_3 = \Delta(1/3, 2/3, (3+\delta i)/6)$,

and $\Delta_4 = \Delta(1/3, 2/3, (6-\delta i)/12)$.

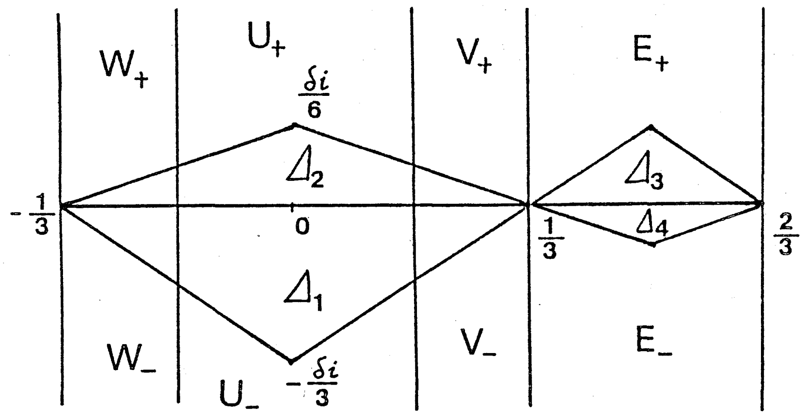


Fig.3.3. Triangles Δ_1, Δ_2, Δ_3, and Δ_4.

We see that

$\Delta_1 \subset U_- \cup V_- \cup W_-$, $\Delta_2 \subset U_+ \cup V_+ \cup W_+$,

$\Delta_3 \subset E_+$, and $\Delta_4 \subset E_-$,

and that

$\tilde{f}(\Delta_3) = \Delta_1$, $\tilde{f}(\Delta_4) = \Delta_1$.

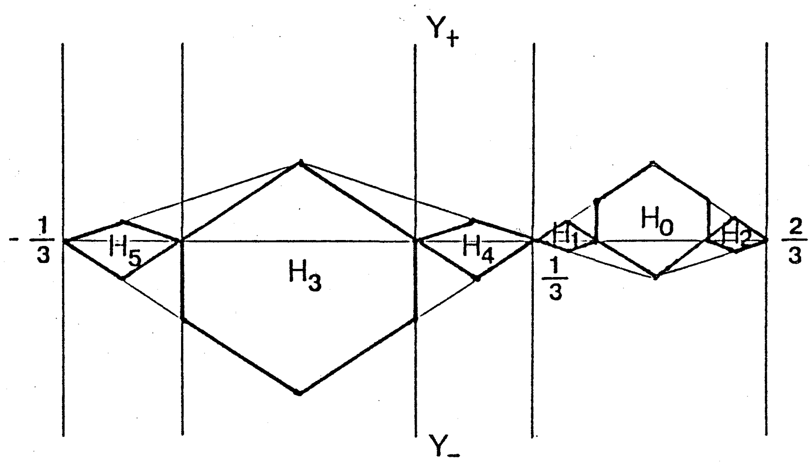


Fig.3.4. Hexagones H_0, H_3 and quadrilaterals H_1, H_2, H_4, H_5.

Let H_0 be the closed hexagone region with vertices
$(3+\delta i)/6$, $(5+\delta i)/12$, $5/12$, $(6-\delta i)/12$, $7/12$, $(7+\delta i)/12$,
and let H_1, H_2 be quadrilaterals defined by

$$H_1 = \Delta(1/3,5/12,(9+\delta i)/24) \cup \Delta(1/3,5/12,(18-\delta i)/48),$$
$$H_2 = \Delta(7/12,2/3,(15+\delta i)/24) \cup \Delta(7/12,2/3,(30-\delta i)/48).$$

Further, let $H_3 = \tilde{f}(H_0)$, $H_4 = \tilde{f}(H_1)$ and $H_5 = \tilde{f}(H_2)$. We see that

$$H_0 \cup H_1 \cup H_2 \subset \Delta_3 \cup \Delta_4 \subset E,$$
$$H_3 \subset (\Delta_1 \cup \Delta_2) \cap U,$$
$$H_4 \subset (\Delta_1 \cup \Delta_2) \cap V,$$
$$H_5 \subset (\Delta_1 \cup \Delta_2) \cap W,$$

and $\tilde{f}(H_4) = \tilde{f}(H_5) = \Delta_3 \cup \Delta_4$.
If we set $H = H_0 \cup \ldots \cup H_5$ and $Y = \bar{L} - H$, then Y has two connected components Y_ν $\{\nu\infty i\}$, $\nu = \pm 1$. We see that

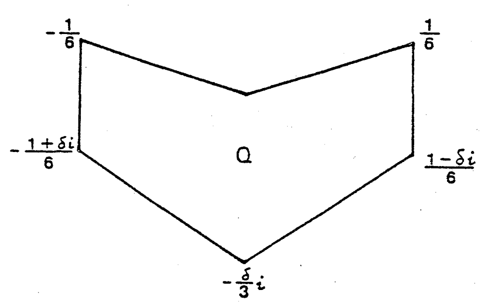
$$\tilde{f}(Y_\nu) \subset Y_{-\nu},\ \nu = \pm 1$$

and $H \supset \tilde{f}^{-1}(H)$.
Note that Y_ν is included in the immediate super-attractive basin $A^*(\nu\infty i)$. Moreover, in the neighborhood of fixed points 1/3 and 2/3, rays $\partial H_1 \cup \partial H_4$ and $\partial H_2 \cup \partial H_5$ are invariant under $\tilde{f}$.

Let Q denote the polygone

$$Q = H_3 - (\Delta_2 \cup \tilde{f}^{-1}(\Delta_2)).$$

This polygone has vertices $-1/6$, $-(1+\delta i)/6$, $-\delta i/3$, $(1-\delta i)/6$, $1/6$, and $-\delta i/12$. We have $Q \subset H - \tilde{f}^{-1}(H)$ and $Q \subset U$.



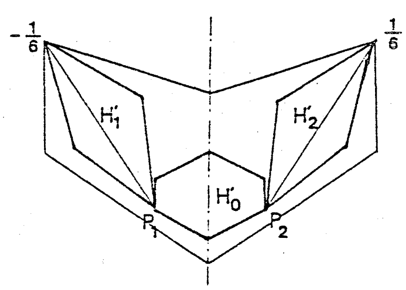


Fig.3.5.(left) Polygone region Q.
Fig.3.6.(right) Hexagone H_0' and quadrilaterals H_1', H_2'.

Proposition 3.1. There exist a hexagone H'_0 and quadrilaterals H'_1, H'_2 such that :

(1) H'_k is affine conformal to H_k for $k = 0,1,2$;

(2) $H' \subset Q$, where $H' = H'_0 \cup H'_1 \cup H'_2$;

(3) $H'_0 \cap \partial Q = \phi$, $H'_1 \cap \partial Q = \{-1/6\}$, $H'_2 \cap \partial Q = \{1/6\}$;

(4) H'_0 intersects H'_k at a single point, say P_k, for $k = 1,2$;

(5) $H'_1 \quad H'_2 = \phi$;

(6) H' is mirror symmetric with respect to the imaginary axis;

(7) there exists an orientation preserving homeomorphism

$$g_1 : H' \to H_0 \cup H_1 \cup H_2,$$

which is affine conformal on each piece H'_k, $k = 1,2$, with

$$g_1(-1/6) = 1/3,\ g_1(1/6) = 2/3,\ g_1(P_1) = 5/12,\ g_1(P_2) = 7/12.$$

The proof is elementary and is omitted.

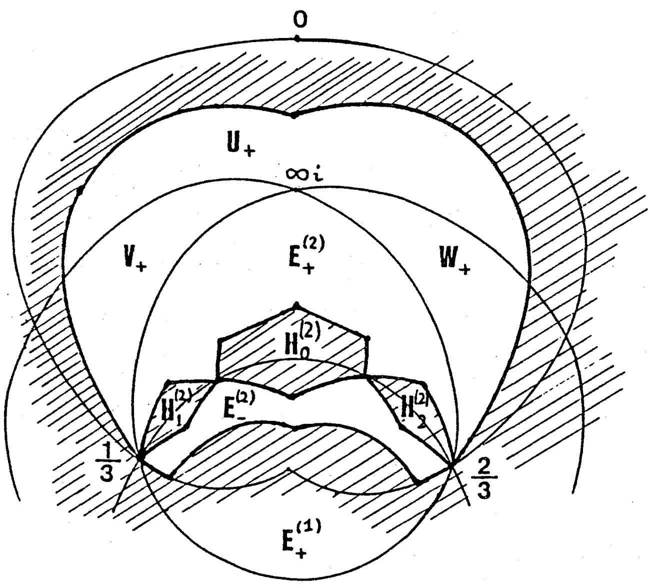


Fig. 3.9. Mapping g_1.

Let H'_0, H'_1, H'_2 Q be as in the proposition above and let Q_1 and Q_2 denote the components of $Q - H'$. Now, let us extend the mapping $g_0 : X - \mathrm{int}(U\_) \to X$ to a quasi-regular map $g : X \to X$ as follows :

$$g(z) = g_0(z) \qquad \text{for} \quad z \in X - \mathrm{int}(U\_),$$

$$g(z) = \tilde{f}(z) \qquad \text{for} \quad z \in \tilde{f}^{-1}(\Delta_2) \cap U_-,$$

$$g(z) = (z+\tfrac{1}{2})^{(2)} \qquad \text{for} \quad z \in U_- - H_3,$$

and $\qquad g(z) = g_1(z)^{(2)} \qquad \text{for} \quad z \in H'.$

It still remains to define the mapping g on regions Q_1 and Q_2. Recall that the conformal structure of X is the same as that of $\bar{L}$ except at the fixed points $\frac{1}{3}$ and $\frac{2}{3}$.

Proposition 3.2. There exist quasiconformal homeomorphisms

$h_1 : \bar{Q}_1 \to (W_+ \cup U_+ \cup V_+ \cup E_+^{(2)}) - \text{int}(\Delta_2 \cup H_0^{(2)} \cup H_1^{(2)} \cup H_2^{(2)})$,

and

$h_2 : \bar{Q}_2 \to$ closure of $E_-^{(2)} - (g(U_- - H_3) \cup H_0^{(2)} \cup H_1^{(2)} \cup H_2^{(2)})$,

mirror symmetric with respect to the imaginary axis, such that $h_1 = g$ on ∂Q_1 and $h_2 = g$ on ∂Q_2.

Proof. Note that g is piecewise linear on ∂Q_1 and ∂Q_2 in our coordinate. Define h_1 and h_2 in the neighborhoods of the points $\frac{1}{3}$, $\frac{2}{3}$, P_1, and P_2 by affine maps so that they agree with g along the boundaries ∂Q_1 and ∂Q_2 in the neighborhoods of these points. Then extend it to diffeomorphisms on the rest of the regions $\bar{Q}_1$ and $\bar{Q}_2$. By the compactness argument, the obtained homeomorphisms h_1 and h_2 are quasiconformal. This construction can be done respecting the mirror symmetry.

We take these quasiconformal maps to define g on these regions.

Proposition 3.3. The map $g : X \to X$ is a quasi-regular map.

Proof. Let σ_1 denote the conformal structure of the Riemann surface X. Let $\sigma_2 = g^*\sigma_1$ be the pull-back by g of the conformal structure. Let X_2 be the Riemann surface X with conformal structure σ_2. Then the identity map $id_2 : X \to X_2$ is a quasiconformal homeomorphism and $g_2 : X_2 \to X$ is conformal. Hence $g = g_2 \circ id_2$ is a quasi-regular mapping.

Theorem 3.4. There exists a quasiconformal homeomorphism $\psi : X \to \bar{\mathbb{C}}$ such that $F = \psi \circ g \circ \psi^{-1}$ is a rational function of degree two of Blaschke's type.

Proof. Observe that $g : X \to X$ is conformal on $X - (\bar{Q}_1 \cup \bar{Q}_2)$. Let

$$Y_1 = \mathrm{int}(U_- \cup V_- \cup W_- \cup (E_-^{(1)} - H^{(1)}) \cup \{-\infty i^{(1)}\}),$$

$$Y_2 = \mathrm{int}((E_+^{(1)} - H^{(1)}) \cup \{+\infty i^{(1)}\} \cup (E_-^{(2)} - H^{(2)})),$$

and $Y_3 = \mathrm{int}(U_+ \cup V_+ \cup W_+ \cup (E_+^{(2)} - H^{(2)}) \cup \{+\infty i^{(2)}\})$.

We see that g is conformal on $Y_1 \cup Y_2 \cup Y_3$ and

$g(Y_1) \subset Y_2$, $g(Y_2) = Y_3$, $g(Y_3) \subset Y_1$.

Observe that $g(\mathrm{int}(Q_1)) \subset Y_3$ and $g(\mathrm{int}(Q_2)) \subset Y_2$.

Let $Y = Y_1 \cup Y_2 \cup Y_3$. Then we have

$g(Y) \subset Y$,

g is analytic on Y,

$\frac{\partial g}{\partial z} = 0$ on $X - g^{-1}(Y)$.

Hence we can apply Lemma 2.1 to obtain a quasiconformal homeomorphism $\psi : X \to \bar{\mathbb{C}}$ such that $F = \psi \circ g \circ \psi^{-1}$ is analytic on $\bar{\mathbb{C}}$. As we have respected the mirror symmetry in our construction of g, we can choose the quasiconformal map ψ so that F is "mirror symmetric" with respect to the unit circle.

Theorem 3.5. The Blaschke's function F obtained in the preceding theorem has a doubly super-attractive cycle of period three.

§4. JULIA SET AND SELF-SIMILAR LATTICE

In this section, we describe a "self-similar lattice" and a dynamical system on the lattice. This dynamical system is topologically conjugate to the restriction to its Julia set of the Blaschke's function constructed in the preceding section.

The first generation of the lattice, L_1, is a lattice composed of 3 bonds and 2 sites. The two sites corresponds to the origin, 0, and the infinity, ∞, in the Riemann sphere. Denote the three bonds by A_0, A_1, A_2 (Fig.4.1).

To get the second generation of lattice, L_2, we replace each bond A_i, i=0,1,2, by a set of four bonds (Fig.4.2).

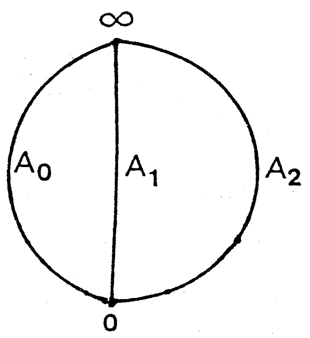
The replacement of bonds is done iteratively to obtain lattices L_3, L_4,...(Fig.4.3).

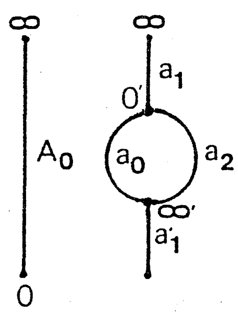
We obtain a self-similar lattice L_∞ as the limit of this procedure. The topology of L_∞ is given naturally by the projective limit topology. A continuous map $u : L_\infty \to L_\infty$ is defined by

$u(\infty) = \infty$, $u(0) = 0$, $u(A_1) = A_2$, $u(A_2) = A_0$,

$u(0') = 0$, $u(\infty') = \infty$, $u(a_0) = A_0$, $u(a_1) = u(a_1') = A_1$, $u(a_2) = A_2$.

The conjugacy map $\chi : L_\infty \to J_f$ is defined by using our quasiconformal map ψ.





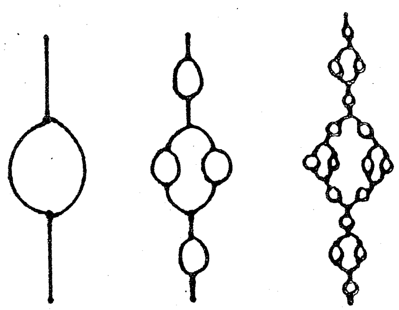


Fig.4.1.(left) First generation L_1 of self-similar lattice.

Fig.4.2.(middle) Bond A_i is replaced by four bonds. We denote the four bonds that replace A_0 by a_0, a_1, a_1', and a_2 as in the figure.

Fig.4.3.(right) Successive replacement of bonds.

Acknowledgments

The author would like to express his gratitude to M.Taniguchi and

M.Shishikura for their help in the preparation and the revision of this article. M.Shishikura provided a brief proof of theorem 1.5, and he helped the author to improve theorem 1.6.

References

[1] L.Ahlfors, Lectures on Quasiconformal Mappings, Van Nostrand, 1966.

[2] L.Ahlfors and L.Bers, Riemann's mapping theorem for variable metrics, Ann. Math. **72**(1960),385-404.

[3] L.Bers and H.L.Royden, Holomorphic families of injections, Acta Math.,**157**(1986), 259-286.

[4] B.Branner and A.Douady, Surgery on complex polynomials, preprint.

[5] P.Blanchard, Complex analytic dynamics on the Riemann sphere, Bull. Amer. Math. Soc., **11**(1984),85-141.

[6] H.Brolin, Invariant sets under iteration of rational functions, Ark. Math.,**6**(1965),103-144.

[7] P.Cvitanović, M.H.Jensen, L.P.Cadanoff, and I.Procaccia, Circle maps in the complex plane, Fractals in Physics, eds. L.Pietronero and E. Tosatti, North Holland,1986, 439-445.

[8] B.Derrida, Real space renormalization and Julia sets in statistical mechanics, Chaotic Dynamics and Fractals, eds. M.F.Barnsley and S.G. Demko, Academic Press, 1986, 229-242.

[9] A.Douady, Systèmes dynamiques holomorphes, Séminaire N.Bourbaki, n° 599, 1982/1983.

[10] A.Douady and J.Hubbard, On the dynamics of polynomial-like mappings, Ann. Scient. Ec. Norm. Sup., 4^e Sér, **18**(1985), 287-343.

[11] P.Fatou, Sur les équations fonctionelles, Bull. Soc. Math. Fr., **47**(1919),161-271, **48**(1920),33-94,208-304.

[12] L.Glass and J.Belain, Continuation of Arnold tongues in mathematical models of periodically forced biological oscillation, Lecture Notes in Biomathematics n°66, Nonlinear Oscillations in Biology and Chemistry.

[13] G.Julia, Mémoir sur l'iteration des fonctions rationnelles, J.Math. Pures et Appl., **8**(1918), 47-245.

[14] R.S.Mackay and C.Tresser, Transition to topological chaos for circle maps, Physica 19D(1986), 206-237.

[15] R.Mañe, P.Sad and D.Sullivan, On the dynamics of rational maps, Ann.Ecole Norm. Sup., to appear.

[16] B.Rodin, Behavior of the Riemann Mapping Function under Complex Analytic Deformations of the Domain, Complex Variables, **5**(1986), 189-195.

[17] M.Shishikura, On the quasiconformal surgery of rational functions, Ann. Scient. Ec. Norm. Sup., 4^e Sér, **20**(1987), 1-29.

[18] D.P.Sullivan and W.P.Thurston, Extending holomorphic motions, Acta Math. **157**(1986),243-257.

[19] S.Ushiki, H.-O.Peitgen and F.v.Haesler, Hyperbolic components of rational fractions $\lambda(z+1/z)$, The Theory of Dynamical Systems and Its Applications to Non-linear Problems, World Sci. Publ., 1984, 61-70.

[20] S.Ushiki, Chaotic phenomena and fractal objects in numerical analysis, Patterns and Waves - Qualitative Analysis of Nonlinear Differential Equations, Studies in Mathematics and Its Applications 18, North Holland, 1986, 221-258.

[21] S.Ushiki, A surgery on Blaschke's function, preprint.

[22] S.E.Warschawski, Recent results in numerical methods of conformal mapping, Proceedings of Symposia in Applied Mathematics, **6**, American Mathematical Society, 1956.

[23] D.Zeitlin, Behavior of conformal maps under analytic deformations of the domain, Doctoral Thesis, University of Minnesota, 1957.