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Symbol Theory of Microlocal Operators

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Let $M \subset \mathbb{R}^n$ be an open neighborhood of the origin, and X be its complex neighborhood. We denote by x the variables of M or of X, and by ξ the dual variables of x. Thus we use the same letter for both real and complex variables, if confusion is not likely. We identify M with the diagonal set in $M \times M$. The sheaf $\mathcal{L} = \mathcal{L}_M$ of microlocal operators is defined on $\sqrt{-1}S^*M = \sqrt{-1}S_M^*(M \times M)$ by

$$\mathcal{L} = \mathcal{H}^0_{\sqrt{-1}S_M^*(M\times M)}(\mathcal{C}_{M\times M} \otimes \mathcal{V}_M)$$

(See [3] and [8]). Here $\mathcal{C}_{M\times M}$ denotes the sheaf of microfunctions on $M\times M$, and $\mathcal{V}_M$ that of densities on M with analytic coefficients. The sheaf $\mathcal{L}$ used to be defined to be the inverse image under the antipodal map of the above one, but we prefer the present definition. A section of $\mathcal{L}$ is called a microlocal operator on $\sqrt{-1}S^*M$ (or on M). A microlocal operator acts on microfunctions, without increasing the support. This property is called the microlocal property, and one may understand that the notion of microlocal operator is the most general one possessing this microlocal property.

There are several subclasses of microlocal operators which are very familiar to us. We denote by $\mathcal{E}^\infty = \mathcal{E}^\infty_M$ (resp. $\mathcal{E}^{\mathbb{R}} = \mathcal{E}^{\mathbb{R}}_M$) the sheaf of microdifferential operators on M (resp. holomorphic microlocal operators on M) defined by [3] and [8] (resp. [4]). We do not give the definitions of these sheaves, but later we will give a description of them from symbol theoretical point of view, and then their meanings will become clear. Note that we have

$$\mathcal{E}^{\infty} \subset \mathcal{E}^{\mathbb{R}} \subset \mathcal{L}.$$

Thus far the symbol theory has been known only for these special subclasses of microlocal operators, and our purpose is to extend it for the general microlocal operators. The details will be given in [9].

To state the main theorem, we give some preliminaries:

Definition 1. Let $r > 0$ be small enough. A continuous function $\lambda(t)$ defined on $0 < t < r$ is called a scaling function if

(i) $\lim_{t \to +0} \lambda(t)/t = 0$,

(ii) $t < t' \Rightarrow \lambda(t)/t < \lambda(t')/t'$.

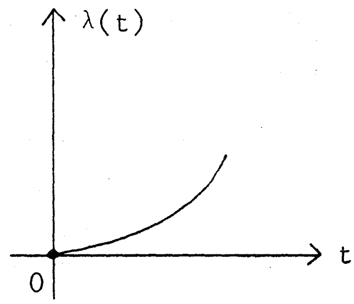

figure: the graph of a scaling function.

Remark. Let $\lambda(t)$ be a scaling function. It is easy to see that we have

(i)' $\lim_{t \to +0} \lambda(t) = 0$,

(ii)' $t < t' \Rightarrow \lambda(t) < \lambda(t')$.

It is easy to see that $\lambda(t)$ is positive definite, and we define $\lambda(0) = 0$ if $t = 0$.

Example. (i) $\lambda(t) = mt^{i}$, $m > 0$, $i > 1$.

(ii) $\lambda(t) = mt(-\log t)^{i}$, $m, i > 0$.

Definition 2. Let $\mathring{x}^* = (0;0,\ldots,0,\sqrt{-1}) \in \sqrt{-1}S^*M$. Let $\lambda(t)$ and $\mu(t)$ be two scaling functions. We denote by $\mathcal{S}_{\lambda,\mu} = (\mathcal{S}_{\lambda,\mu})_{\mathring{x}^*}$ the space of holomorphic functions $a(x,\xi)$ defined on some conical complex neighborhood V of $\mathring{x}^*$ in T^*X such that for any $\varepsilon > 0$ there exists some $C_\varepsilon > 0$ satisfying

$$(1) \qquad |a(x,\xi)| \leqq C_\varepsilon \exp\{(\lambda(|\mathrm{Im}\, x|) + \mu(|\mathrm{Re}\,\xi|/\mathrm{Im}\,\xi_n) + \varepsilon)\mathrm{Im}\,\xi_n\}$$

on V. We define $\mathcal{S} = \mathcal{S}_{\mathring{x}^*}$ by $\mathcal{S} = \bigcup_{\lambda,\mu} \mathcal{S}_{\lambda,\mu}$. We denote by $\mathcal{N} = \mathcal{N}_{\mathring{x}^*}$ the space of $a(x,\xi) \in \mathcal{S}$ such that there exist some $\varepsilon > 0$ and $C > 0$ satisfying

$$(2) \qquad |a(x,\xi)| \leqq C\exp(-\mathrm{Im}\,\xi_n)$$

on some conical complex neighborhood of $\mathring{x}^*$.

Example. (i) $\exp(\sqrt{-1}\xi_j{}^2/\xi_n) \in \mathcal{S}_{\mathring{x}^*}$, $1 \leqq j \leqq n-1$.
(ii) $\exp(\sqrt{-1}x_j{}^2\cdot\xi_n) \in \mathcal{S}_{\mathring{x}^*}$, $1 \leqq j \leqq n$.

Let $x, x' \in M$, and let $A = u(x,x')dx' \in \mathcal{L}_{\mathring{x}^*}$, $u(x,x')$ being the kernel function of A. We can define the Fourier transform $\hat{u}(x,\xi)$ of $u(x,x')$ by

$$(3) \qquad \hat{u}(x,\xi) = \int e^{-(x-x')\xi}u(x,x')dx'.$$

We do not give the precise meaning of this integral, but one may understand that it is defined in a natural way as a microfunction. (3) has some ambiguity, but one can prove

Proposition 3. $u(x,\xi) \in \mathcal{S}$.

If one neglects an element of $\mathcal{N}_{\overset{\circ}{X}^*}$, then $\hat{u}(x,\xi)$ is defined with no ambiguity. Thus the following map is well-defined:

$$(4) \qquad \sigma : \mathcal{L}_{\overset{\circ}{X}^*} \ni A = u(x,x')dx' \longmapsto \sigma(A) = \hat{u}(x,\xi) \in \mathcal{S}_{\overset{\circ}{X}^*}/\mathcal{N}_{\overset{\circ}{X}^*} .$$

<u>Theorem 4</u>. (4) is an isomorphism.

<u>Definition 5</u>. We call $\sigma(A)(x,\xi)$ the <u>symbol function</u> of A.

<u>Remark</u>. Let us resume the symbol theory already known. We denote by $\mathcal{S}_1 = (\mathcal{S}_1)_{\overset{\circ}{X}^*} \subset \mathcal{S}_{\overset{\circ}{X}^*}$ the space of all $a(x,\xi) \in \mathcal{S}_{\overset{\circ}{X}^*}$ which satisfy: For any $\varepsilon > 0$ there exists some $C_\varepsilon > 0$ such that (1) is valid with $\lambda(t) = \mu(t) = 0$. i.e., $\mathcal{S}_1$ is the space of infra-exponential symbol functions. We denote by $\mathcal{S}_2 = (\mathcal{S}_2)_{\overset{\circ}{X}^*}$ the space of all $a(x,\xi) \in (\mathcal{S}_1)_{\overset{\circ}{X}^*}$ which have asymptotic expansions $a(x,\xi) \sim \sum_{j\in\mathbb{Z}} a_j(x,\xi)$ where each $a_j(x,\xi)$ is homogeneous in ξ of degree j (See [8] for the precise definition). In [3] and [8] it is proved that we have

$$\mathcal{E}^{\infty}_{\overset{\circ}{X}^*} \cong (\mathcal{S}_2)_{\overset{\circ}{X}^*}/\mathcal{N}_{\overset{\circ}{X}^*} ,$$

and in [1] and [5] that

$$\mathcal{E}^{\mathbb{R}}_{\overset{\circ}{X}^*} \cong (\mathcal{S}_1)_{\overset{\circ}{X}^*}/\mathcal{N}_{\overset{\circ}{X}^*} .$$

Thus we have obtained

$$\begin{array}{ccccc} \mathcal{E}^{\infty}_{\overset{\circ}{X}^*} & \subset & \mathcal{E}^{\mathbb{R}}_{\overset{\circ}{X}^*} & \subset & \mathcal{L}_{\overset{\circ}{X}^*} \\ \wr\| & & \wr\| & & \wr\| \\ \mathcal{S}_2/\mathcal{N} & \subset & \mathcal{S}_1/\mathcal{N} & \subset & \mathcal{S}/\mathcal{N} . \end{array}$$

We next want to give the symbol formulae for adjoint operators and composite operators of microlocal operators. There arises one problem then, and we first explain about it. Let $A \in \mathcal{L}_{\mathcal{O}^*_X}$ and assume that $\sigma(A) \in \mathcal{S}_{\lambda,\mu}$ for some scaling functions $\lambda(t)$ and $\mu(t)$. The question is: Can we prove

$$(5) \qquad \sigma(A^*)(x,-\xi) \sim \sum_{\alpha} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_x^{\alpha}\partial_{\xi}^{\alpha}\sigma(A)(x,\xi)$$

in some sense ? There is one more question concerning the composite operators. If one wants to obtain (5), one needs to estimate the derivatives $\partial_x^{\alpha}\partial_{\xi}^{\beta}\sigma(A)(x,\xi), \alpha,\beta \in \mathbb{Z}_+^n$. We can prove the foolowing

<u>Proposition 6</u>. Let $C_0 > 0$. We define $T(k) \subset \mathbb{C}^n \times \mathbb{C}^n$, $k \in \mathbb{Z}_+$, by

$$\begin{aligned} T(k) = \{(x,\xi) \in \mathbb{C}^n \times \mathbb{C}^n;\ & C_0|x| < 1, \\ & \operatorname{Im} \xi_n > C_0|\operatorname{Im} \xi_j|,\ 1 \leq j \leq n-1, \\ & \operatorname{Im} \xi_n > C_0|\operatorname{Re} \xi_j|,\ 1 \leq j \leq n,\ \operatorname{Im} \xi_n > C_0 k\} . \end{aligned}$$

Assume that C_0 and C_1 are large enough, and that $k > |\alpha|, |\beta|$. Let $\lambda(t)$ and $\mu(t)$ be the above scaling functions. For any $\varepsilon > 0$ there exists some $C_\varepsilon > 0$ such that

$$\begin{aligned} (6) \qquad |\partial_x^{\alpha}\partial_{\xi}^{\beta}\sigma(A)(x,\xi)| \leq\ & C_\varepsilon \alpha!\beta! \\ & \times \exp\{(\lambda(C_1|\operatorname{Im} x|) + \mu(C_1|\operatorname{Re} \xi|/\operatorname{Im} \xi_n) + \varepsilon)\operatorname{Im} \xi_n\} \\ & \times (C_1/\lambda^{-1}(|\alpha|/\operatorname{Im} \xi_n))^{|\alpha|}(C_1/\mu^{-1}(|\beta|/\operatorname{Im} \xi_n)\operatorname{Im} \xi_n)^{|\beta|} \end{aligned}$$

on $T(k)$.

This is a direct corollary of Cauchy integration theorem. See [9] for the proof. (6) looks more familiar in the following special case: Assume that $\lambda(t) = m_1 t^i$, $\mu(t) = m_2 t^j$, $m_1, m_2 > 0$, $i, j > 1$, and that $C_\varepsilon = C$ does not depend on ε. Then we have

$$\lambda^{-1}(t) = (t/m_1)^{1/i}, \quad \mu^{-1}(t) = (t/m_2)^{1/j}$$

($\lambda^{-1}(t)$ denotes the inverse function of $\lambda(t)$). If $(x,\xi) \in T(k) \cap \sqrt{-1}T^*M$ (and thus $\operatorname{Im} x = \operatorname{Re} \xi = 0$), (6) means

(6)' $$|\partial_x^\alpha \partial_\xi^\beta \sigma(A)(x,\xi)|$$

$$\leqq C \cdot C_1^{|\alpha|+|\beta|} m_1^{|\alpha|/i} m_2^{|\beta|/j} \alpha!^{(i-1)/1} \beta!^{(j-1)/j}$$

$$\times (\operatorname{Im} \xi_n)^{|\alpha|/i-(j-1)|\beta|/j} .$$

It follows that $\sigma(A)(x,\xi) \in S^0_{\rho,\delta}(\mathbb{R}^n)$ with $\rho = 1 - 1/j$, $\delta = 1/i$. Here $S^0_{\rho,\delta}$ denotes the symbol space introduced by [2]. Our theory shows some similarity to the symbol theory of $S^m_{\rho,\delta}$ in distribution theory, or, more generally, to the symbol theory of $S^m_{\nu,\rho,\delta}$, with some "basic weight function" $\nu(\xi)$, introduced by H. Kumanogo. The auther does not know whether this similarity has some deeper meaning or not, but concerning the symbol calculation, there arises the following problem.

Let A be a pseudodifferential operator in distribution theory, and let $\sigma(A)(x,\xi)$, the complete symbol of A, belong to $S^m_{\rho,\delta}$, $m \in \mathbb{Z}$, $0 < \rho, \delta < 1$. If one wants to obtain a symbol formula (5) for A^*, then one needs to assume $\rho > \delta$ (or at least $\rho \geqq \delta$). If $\rho < \delta$, then one cannot expect to obtain (5), since the right-hand side of (5) becomes larger and larger as $|\alpha| \to +\infty$, and thus this asymptotic expansion does not have any meaning. Note that we have used a notation standard in hyperfunction theory, and the asymptotic expansion (5) should be given in a slightly different form, if one follows the convention standard in distribution theory. Anyway an analogous problem occurs in our symbol theory again, and it seems inevitable to assume some condition on $\lambda(t)$ and $\mu(t)$, if one wants to prove (5) for A^*, with $\sigma(A) \in \mathcal{S}_{\lambda,\mu}$.

Let $\lambda(t)$ and $\mu(t)$ be two scaling functions. Let $C_0, C_1 > 0$ be two given constants. We introduce the following

Condition C_0, C_1. $0 < t < 1/C_0 \Rightarrow \lambda^{-1}(t)\mu^{-1}(t) \geqq C_1 t$.

Example. Let $\lambda(t) = m_1 t^i$, $\mu(t) = m_2 t^j$, $m_1, m_2 > 0$, $i, j > 1$. Then Condition C_0, C_1 is satisfied if, and only if, either

$$1/i + 1/j < 1$$

or

$$1/i + 1/j = 1, \; m_1^{-1/i} m_2^{-1/j} \geqq C_1.$$

Let C_0, C_1 be large enough, and assume that $\lambda(t)$ and $\mu(t)$ satisfy Condition C_0, C_1. If $\sigma(A) \in \mathcal{S}_{\lambda,\mu}$, then we have the desired asymptotic expansion (5). More precisely, we have the following result: We define $B_j(x,-\xi), j \in \mathbb{Z}_+$, by

$$B_j(x,-\xi) = \sum_{|\alpha|=j} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_x^{\alpha} \partial_\xi^{\alpha} \sigma(A)(x,\xi),$$

and consider the following power series:

(7) $$\sum_{j=0}^{\infty} \frac{1}{(2\pi\sqrt{-1})^n} \int e^{(x-x')\xi} B_j(x,\xi)d\xi.$$

Here each integral is calculated on the following domain

$$\{\xi \in \sqrt{-1}\mathbb{R}^n;\ \mathrm{Im}\ \xi_n \leqq -C_0|\mathrm{Im}\ \xi_k|,\ 1 \leqq k \leqq n-1,\ \mathrm{Im}\ \xi_n \leqq -C_0(j+1)\}$$

with the corresponding number $j \in \mathbb{Z}_+$. Then we have the following

Theorem 7. Under the above assumptions, let $u(x,x')$ be the kernel function of A: $A = u(x,x')dx'$. Then the power series (7) converges and it becomes the defining function of $u(x',x)$.

Since we have $A^* = u(x',x)dx'$ by definition, this is the desired result.

As for the composite operator, we have the following

Theorem 8. Let $A_i = u_i(x,x')dx' \in \mathcal{L}_{\Omega^*_X}$, $i = 1,2$. Let $\sigma(A_i) \in \mathcal{S}_{\lambda_i,\mu_i}$, with some scaling functions $\lambda_i(t)$, $\mu_i(t)$, $i = 1,2$. Assume that $\mu_1(t)$ and $\lambda_2(t)$ satisfy Condition C_0, C_1, where C_0 and C_1 are large enough. Then we have

(8) $$\sigma(A_1A_2)(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_\xi^{\alpha}\sigma(A_1)(x,\xi)\partial_x^{\alpha}\sigma(A_2)(x,\xi)$$

We do not explain the precise meaning of (8), since it is similar to that of (5).

Remark. If A is a holomorphic microlocal operator, then Theorem 7 always applies. If one of A_1 and A_2 is a holomorphic microlocal operator, then Theorem 8 always applies.

Finally we explain the relationship of our theory and that of

Fourier integral operators with complx phase functions[7]. Assume that $a(x,\xi)$ is written in the form

$$a(x,\xi) = e^{\varphi(x,\xi)} a_1(x,\xi), \tag{9}$$

where the amplitude function $a_1(x,\xi)$ belongs to $(\mathcal{S}_1)_{\mathring{x}^*}$, and the phase function $\varphi(t)$ satisfies

$$|\mathrm{Re}\ \varphi(x,\xi)| \leqq (\lambda(|\mathrm{Im}\ x|) + \mu(|\mathrm{Re}\ \xi|/\mathrm{Im}\ \xi_n))\mathrm{Im}\ \xi_n$$

with some scaling functions $\lambda(t)$ and $\mu(t)$. Then we have $a(x,\xi) \in \mathcal{S}_{\mathring{x}^*}$. On the other hand, $a(x,\xi)$ may be regarded as the symbol of a Fourier integral operator(more precisely, a pseudodifferential operator) with a complex phase function. Our theory may be regarded as a generalization of such a theory, since we do not assume that $a(x,\xi)$ is written in the form (9). We only assume that it satisfies an estimate of the form (1)

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Symbol Theory of Microlocal Operators

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序文．　この論説文では，佐藤超函数論におけるいわゆる「擬微分作用素」として，最も一般的な概念である超局所作用素をとりあげる．先ず，超局所作用素の表象を定義し，簡単な特徴付けを与える．超局所作用素といえば，これまではその正体・性質が十分よく理解されていたわけではなかったが，表象を通じて，意外に簡単にその正体を把握できるということがわかる．次に，共役作用素や合成作用素の表象公式を与える．その際，C^∞理論との興味深い対比が見出される．佐藤超函数論における擬微分作用素というものは，これまではなんとなく$S^m_{1,0}$的なものだと信じられていた．しかし実はそうではなくて，一般的な$S^m_{\rho,\delta}$空間とも関係があるということを強調しておく．