数理解析研究所講究録
第 644 巻 1988 年 30-43


# A note on Schmidt's built-up systems of fundamental sequences

Noriya Kadota and Kiwamu Aoyama
角田法也 青山究

Introduction. Let $\langle F_\alpha \rangle_{\alpha \in \Delta}$ be a transfinite sequence of number theoretic functions indexed by an initial segment $\Delta$ of the second number class which satisfies the following conditions:

(a) $F_0$ is strictly increasing,

(b) if $F_\alpha$ is strictly increasing, $F_{\alpha+1}$ is also strictly increasing, $F_\alpha(0) \leq F_{\alpha+1}(0)$ and $F_\alpha(x) < F_{\alpha+1}(x)$ for $0 < x < \omega$,

(c) $F_\alpha(x) = F_{\alpha[x]}(x)$ if $\alpha$ is a limit ordinal, where $\langle \alpha[x] \rangle_{x<\omega}$ is a fundamental sequence for $\alpha$.

Schmidt[3] introduced the concept of built-up systems of fundamental sequences, and showed that, for the above sequence $\langle F_\alpha \rangle_{\alpha \in \Delta}$, each $F_\alpha$ is strictly increasing if the system of fundamental sequences used is built-up. However there are some standard systems of fundamental sequences in literatures, e.g. Ketonen and Solovay[1], which are not built-up in Schmidt's sense, but which determine a sequence of strictly increasing functions.

The purpose of this note is to extend the concept of built-up systems so that it can be applicable to wider classes of systems of sequences of ordinals.

In §1 we define (n)-built-up systems and quasi-(n)-built-up systems of sequences of ordinals. In §2 we show a theorem on a relation between (n)-built-up systems and $\langle F_\alpha \rangle_{\alpha \in \Delta}$, which corresponds to Theorem 1 in [3], and a theorem on a relation between quasi-(n)-built-up systems and a sequence of number theoretic functions $\langle H_\alpha \rangle_{\alpha \in \Delta}$. In §3, we give an example of (1)-built-up system of fundamental sequences for $\Gamma_0$. Finally, in §4, we extend the results in Schmidt[4], by using quasi-(n)-built-up systems of sequences of ordinals.

Authors would like to express their deep thanks to Prof. H. Ono for valuable discussions, Prof. A. Nakamura for encouragement on this work, Prof. T. Uesu and Dr. T. Arai for helpful comments.

## §1. Preliminaries

Let $\Delta$ be an initial segment of second number class. We will use Greek letters $\alpha$, $\beta$, $\gamma$, ... for ordinal numbers in $\Delta$. Let $P:\Delta \longrightarrow \Delta^\omega$ be an assignment of sequences of ordinals for $\Delta$. We shall write $\alpha[i]$ for $(P(\alpha))(i)$ whenever $\alpha \in \Delta$ and $i < \omega$.

If P satisfies the following conditions (A)-(C):

(A) $\alpha[i] = 0$ if $\alpha = 0$ and $i < \omega$,

(B) $\alpha[i] = \beta$ if $\alpha = \beta + 1$ and $i < \omega$,

(C) $\alpha[i] < \alpha$ if $\alpha$ is a limit ordinal and $i < \omega$,

then we call P a <u>system of sequences of ordinals</u> for $\Delta$. Moreover, if a system P of sequences of ordinals satisfies the following conditions $(C)^+$,(D):

$(C)^+$ $\alpha[i] < \alpha[i+1] < \alpha$ if $\alpha$ is a limit ordinal and $i < \omega$,

(D) $\lim_{i<\omega} \alpha[i] = \alpha$ if $\alpha$ is a limit ordinal,

then we call P a <u>system of fundamental sequences</u> for $\Delta$.

In the following, we assume P is a system of sequences of ordinals.

Definition 1.1. Let P be a system of sequences of ordinals for $\Delta$. For each $n < \omega$, $\xrightarrow[n]{1}$, $\xrightarrow[n]{}$, $\underset{n}{\Longrightarrow}$ are defined as follows:

(1) $\alpha \xrightarrow[n]{1} \beta$ iff $0 < \alpha$ and $\alpha[n] = \beta$,

(2) $\alpha \xrightarrow[n]{} \beta$ iff there is a sequence $\gamma_0, \ldots, \gamma_j$ $(0<j<\omega)$ such that $\gamma_0 = \alpha$, $\gamma_j = \beta$ and $\gamma_i \xrightarrow[n]{1} \gamma_{i+1}$ $(0 \leq i < j)$,

(3) $\alpha \underset{n}{\Longrightarrow} \beta$ iff $\alpha \xrightarrow[n]{} \beta$ or $\alpha = \beta$.

For each $n < \omega$, P is (n)-built-up ( and quasi-(n)-built-up ), if $\alpha[i+1] \xrightarrow[n]{} \alpha[i]$ ( and $\alpha[i+1] \underset{n}{\Longrightarrow} \alpha[i]$, respectively ) for each limit ordinal and each $i < \omega$.

Built-up systems in Schmidt's sense[3] is the same as (0)-built-up systems of fundamental sequences in our sense. Ketonen and Solovay[1] introduced the relation $\xrightarrow[n]{}$ for studying a standard system of fundamental sequences for ordinals up to $\varepsilon_0$. Their system is (1)-built-up but not (0)-built-up (cf.Theorem 2.4 of [1]).

Proposition 1.1. Let P be quasi-(s)-built-up. If $s \leq m$, $n \leq m$ and $\alpha \xrightarrow[n]{} \beta$, then $\alpha \xrightarrow[m]{} \beta$.

(Proof) By induction on $\alpha$. Case 1. $\alpha = 0$. This case is trivial because $\neg(0 \xrightarrow[n]{} \beta)$. Case 2. $\alpha=\gamma+1$. If $\alpha \xrightarrow[n]{} \beta$, then $\gamma \underset{n}{\Longrightarrow} \beta$. So $\alpha=\gamma+1 \xrightarrow[m]{1} \gamma$ and $\gamma \underset{m}{\Longrightarrow} \beta$ by ind. hyp. So $\alpha \xrightarrow[m]{} \beta$. Case 3. $\alpha$ is limit. If $\alpha \xrightarrow[n]{} \beta$ then $\alpha[n] \underset{n}{\Longrightarrow} \beta$. Because P is quasi-(s)-built-up, $\alpha[m] \underset{s}{\Longrightarrow} \alpha[n]$. By ind. hyp., $\alpha[m] \underset{m}{\Longrightarrow} \alpha[n]$ and

$\alpha[n] \underset{m}{\Longrightarrow} \beta$. So, $\alpha \xrightarrow[m]{1} \alpha[m] \underset{m}{\Longrightarrow} \alpha[n] \underset{m}{\Longrightarrow} \beta$.

Corollary 1.2. Let $n \leq m$. If P is (n)-built-up ( and quasi-(n)-built-up ), then P is (m)-built-up ( and quasi-(m)-built-up, respectively ).

## §2. (n)-built-up systems and hierarchies of number theoretic functions

We say that a function $F:\omega \longrightarrow \omega$ is strictly increasing after n if $F(x) < F(x+1)$ for $n \leq x < \omega$.

Let P be a system of sequences of ordinals for $\Delta$. Suppose that $\langle F_\alpha \rangle_{\alpha\in\Delta}$ is any sequence of number theoretic functions satisfying the following conditions for $n < \omega$.

$(a)_n$ $F_0$ is strictly increasing after n.

$(b)_n$ If $F_\alpha$ is strictly increasing after n, then $F_{\alpha+1}$ is also strinctly increasing after n, $F_\alpha(n) \leq F_{\alpha+1}(n)$, and $F_\alpha(x) < F_{\alpha+1}(x)$ for $n < x < \omega$.

$(c)_n$ $F_\alpha(x) = F_{\alpha[x]}(x)$ for $n \leq x < \omega$, if $\alpha$ is limit.

Remark that conditions $(a)_0$, $(b)_0$ and $(c)_0$ are the same as (a), (b) and (c) in Introduction.

Example 1. Let the fast growing hierarchy $\langle F_\alpha \rangle_{\alpha\in\Delta}$ define by

$F_0(x) = x+1$, $F_{\alpha+1}(x) = F_\alpha^{x+1}(x)$, where $F_\alpha^i$ is defined by $F_\alpha^0(x)=x$, $F_\alpha^{i+1}(x)=F_\alpha^i(F_\alpha(x))$,

$F_\alpha(x) = F_{\alpha[x]}(x)$ if $\alpha$ is limit.

Then $\langle F_\alpha \rangle_{\alpha\in\Delta}$ satisfies $(a)_n$, $(b)_n$ and $(c)_n$.

<u>Theorem 2.1.</u> If $\langle F_\alpha \rangle_{\alpha\in\Delta}$ satisfies conditions $(a)_n$, $(b)_n$ and $(c)_n$, then the following hold for each $\alpha$, $\beta \in \Delta$.

(1) $\alpha \xrightarrow[n]{} \beta$ implies $F_\beta(n) \le F_\alpha(n)$.

(2) If P is (n+1)-built-up, then

(2.1) $F_\alpha$ is strictly increasing after n,

(2.2) $\alpha \xrightarrow[m]{} \beta$ implies $F_\beta(s) \le F_\alpha(s)$, $F_\beta(x) < F_\alpha(x)$ for $s < x < \omega$, where $s = \max(\, n+1,\, m\,)$.

Moreover, if P is (n)-built-up and $m \le n$, then $\alpha \xrightarrow[m]{} \beta$ implies $F_\beta(n) \le F_\alpha(n)$, $F_\beta(n+1) < F_\alpha(n+1)$.

(Proof) By induction on $\alpha$. <u>Case 1.</u> $\alpha = 0$. (1) holds because $\neg(0 \xrightarrow[n]{} \beta)$. (2.1) is $(a)_n$, (2.2) holds because $\neg(0 \xrightarrow[n]{} \beta)$.

<u>Case 2.</u> $\alpha=\gamma+1$. (1) If $\alpha \xrightarrow[n]{} \beta$ then $\gamma \xRightarrow[n]{} \beta$. $F_\beta(n) \le F_\gamma(n) \le F_\alpha(n)$ by ind.hyp. and $(b)_n$. (2.1) Since $F_\gamma$ is strictly increasing after n, by ind. hyp., so is $F_\alpha$ by $(b)_n$. (2.2) Assume $\alpha \xrightarrow[m]{} \beta$. Then $\gamma \xRightarrow[m]{} \beta$. If $\beta = \gamma$, (2.2) holds by $(b)_n$. Hence, by ind. hyp. (2.2) holds for all $\alpha \xrightarrow[m]{} \beta$.

<u>Case 3.</u> $\alpha$ is limit. (1) If $\alpha \xrightarrow[n]{} \beta$ then $\alpha[n] \xRightarrow[n]{} \beta$. So

$$F_\beta(n) \le F_{\alpha[n]}(n) = F_\alpha(n) \quad \text{by (1) of ind. hyp.}$$

(2) Let P be (n+1)-built-up. (2.1) For $n \le x < \omega$, $\alpha[x+1] \xrightarrow[n+1]{} \alpha[x]$. So,

$$\begin{aligned} F_\alpha(x+1) = F_{\alpha[x+1]}(x+1) &\ge F_{\alpha[x]}(x+1) && \text{by (2.2) of ind.hyp.} \\ &> F_{\alpha[x]}(x) && \text{by (2.1) of ind.hyp.} \\ &= F_\alpha(x). \end{aligned}$$

(2.2) If $\alpha \xrightarrow[m]{} \beta$ then $\alpha[m] \xRightarrow[m]{} \beta$. Let $s = \max(n+1, m)$. Then

$$F_\beta(s) \le F_{\alpha[m]}(s) \le F_{\alpha[s]}(s) = F_\alpha(s)$$

by $\alpha[m] \xRightarrow[n+1]{} \alpha[s]$ and ind.hyp.

$F_\beta(x) \le F_{\alpha[m]}(x) < F_{\alpha[x]}(x) = F_\alpha(x)$

for $s < x < \omega$ by $\alpha[x] \xrightarrow[n+1]{} \alpha[m]$, and ind.hyp. Moreover if P is (n)-built-up and $m \le n$, then $\alpha[n] \underset{n}{\Longrightarrow} \alpha[m]$ and $\alpha[n+1] \xrightarrow[n]{} \alpha[m]$,

$F_\beta(n) \le F_{\alpha[m]}(n) \le F_{\alpha[n]}(n) = F_\alpha(n)$,

$F_\beta(n+1) \le F_{\alpha[m]}(n+1) < F_{\alpha[n+1]}(n+1) = F_\alpha(n+1)$, by ind.hyp.

<u>Corollary 2.2.</u> If $\langle F_\alpha \rangle_{\alpha\in\Delta}$ satisfies (a),(b) and (c) ( i.e. $(a)_0$,$(b)_0$ and $(c)_0$ ), and P is (1)-built-up, then for each $\alpha$, $F_\alpha$ is strictly increasing.

Next, we will introduce another hierarchies $\langle H_\alpha \rangle_{\alpha\in\Delta}$ of number theoretic functions which is constructed by the same way as $\langle F_\alpha \rangle_{\alpha\in\Delta}$ except $(c)_n$ ( i.e. $F_\alpha(x) = F_{\alpha[x]}(x)$ if $\alpha$ is a limit ordinal ).

We fix a system P of sequences of ordinals for $\Delta$, and a function $f:\omega \longrightarrow \omega$ which satisfies $n < f(n)$ and $f(x) < f(x+1)$ for $n \le x < \omega$ ( e.g. $f(x) = x + 1$ ).

Suppose $\langle H_\alpha \rangle_{\alpha\in\Delta}$ is any sequence of number theoretic functions satisfying the following conditions for n.

$(a)_n$ and $(b)_n$ are the same as the case of $\langle F_\alpha \rangle_{\alpha\in\Delta}$,

$(c)^*_n$ $H_\alpha(x) = H_{\alpha[x]}(f(x))$ for $n \le x < \omega$, if $\alpha$ is limit.

<u>Example 2.</u> Let the Hardy hierarchy $\langle H_\alpha \rangle_{\alpha\in\Delta}$ define by

$H_0(x) = x$, $H_\alpha(x) = H_{\alpha[x]}(x+1)$ if $\alpha > 0$.

Then $\langle H_\alpha \rangle_{\alpha\in\Delta}$ satisfies $(a)_n$, $(b)_n$ and $(c)^*_n$ ( where we take x+1 as f(x) ).

We can prove a theorem which is a relation between quasi-(n)-built-up systems and $\langle H_\alpha \rangle_{\alpha \in \Delta}$.

<u>Theorem 2.3.</u> If $\langle H_\alpha \rangle_{\alpha \in \Delta}$ satisfies conditions $(a)_n$, $(b)_n$ and $(c)^*_n$ and P is quasi-(f(n+1))-built-up, then for each $\alpha \in \Delta$,

(1) $H_\alpha$ is strictly increasing after n,

(2) $\alpha \xrightarrow[m]{} \beta$ implies $H_\beta(x) < H_\alpha(x)$ for $\max(n+1,m) \le x < \omega$.
In addition, $\alpha \xrightarrow[n]{} \beta$ implies $H_\beta(n) \le H_\alpha(n)$.
Moreover, if P is quasi-(f(n))-built-up and $m < n$, then $\alpha \xrightarrow[m]{} \beta$ implies $H_\beta(n) \le H_\alpha(n)$.

(Proof) By induction on $\alpha$. Assume that P is quasi-(f(n+1))-built-up. <u>Case 1.</u> $\alpha=0$. (1) is $(a)_n$. (2) holds because $\neg(0 \xrightarrow[m]{} \beta)$.
<u>Case 2.</u> $\alpha=\gamma+1$. (1) Since $H_\gamma$ is strictly increasing after n by ind.hyp., so is $H_\alpha$ by $(b)_n$. (2) Assume $\alpha \xrightarrow[m]{} \beta$. Then $\gamma \xRightarrow[m]{} \beta$. If $\beta = \gamma$, (2) holds by $(b)_n$. Hence, by ind. hyp., (2) holds for all $\alpha \xrightarrow[m]{} \beta$.
<u>Case 3.</u> $\alpha$ is limit. (1) For $n \le x < \omega$, $\alpha[x+1] \xRightarrow[f(n+1)]{} \alpha[x]$, $f(n+1) \le f(x+1)$,

$$\begin{aligned} H_\alpha(x+1) = H_{\alpha[x+1]}(f(x+1)) &\ge H_{\alpha[x]}(f(x+1)) && \text{by (2) of ind.hyp.} \\ &> H_{\alpha[x]}(f(x)) && \text{by (1) of ind.hyp.} \\ &= H_\alpha(x). \end{aligned}$$

(2) If $\alpha \xrightarrow[m]{} \beta$ then $\alpha[m] \xRightarrow[m]{} \beta$. For $\max(n+1,m) \le x < \omega$, by ind. hyp. and $\alpha[x] \xRightarrow[f(n+1)]{} \alpha[m]$, $f(n+1) \le f(x)$, so

$$H_\beta(x) \le H_{\alpha[m]}(x) < H_{\alpha[m]}(f(x)) \le H_{\alpha[x]}(f(x)) = H_\alpha(x).$$

If $\alpha \xrightarrow[n]{} \beta$ then $\alpha[n] \xRightarrow[n]{} \beta$. By ind. hyp.,

$$H_\beta(n) \le H_{\alpha[n]}(n) < H_{\alpha[n]}(f(n)) = H_\alpha(n).$$

Moreover if P is quasi-(f(n))-built-up, $\alpha \xrightarrow[m]{} \beta$ and and $m < n$ then $\alpha[m] \overset{}{\underset{m}{\Longrightarrow}} \beta$ and $\alpha[n] \underset{f(n)}{\Longrightarrow} \alpha[m]$, by ind. hyp.,

$$H_\beta(n) \le H_{\alpha[m]}(n) < H_{\alpha[m]}(f(n)) \le H_{\alpha[n]}(f(n)) = H_\alpha(n).$$

## §3. A (1)-built-up system of fundamental sequences for $\Gamma_0$.

We give a system P of fundamental sequences for $\Gamma_0$ ( for details about $\Gamma_0$ see Schütte[5] ), by modifying the system in §3 of Schmidt[3]. If we restrict P to ordinals below $\varepsilon_0$, P corresponds to a standard system of fundamental sequences below $\varepsilon_0$ ( e.g. Ketonen and Solovay[1] ). Then we can prove that P is (1)-built-up (cf.Theorem 2.4 in [1]).

All ordinals below $\Gamma_0$ can be generated from 0 by the two functions $\nu$ and $\kappa$ defined by

$\nu(\alpha, \beta) = \omega^\alpha + \beta$,

$\kappa(0, \beta) = \varepsilon_\beta$ = the $\beta$-th inaccessible of $\nu$,

for $\gamma > 0$, $\kappa(\gamma, \beta)$ = the $\beta$-th ordinal which is inaccessible for all $\lambda\delta.\kappa(\alpha, \delta)$ such that $\alpha < \gamma$.

If $\gamma < \Gamma_0$, $\gamma$ is a limit ordinal, then there is exactly one pair $(\alpha, \beta) \in {\Gamma_0}^2$ and one $\rho \in \{\kappa, \nu\}$ such that $\alpha, \beta < \gamma$ and $\gamma = \rho(\alpha, \beta)$. We will write $\gamma =_{nf} \rho(\alpha,\beta)$ ( $\rho(\alpha,\beta)$ is the normal form of $\gamma$ ). We define a system P of fundamental sequences for all limit ordinals $\gamma < \Gamma_0$ by induction on $\gamma$.

(1) $\gamma =_{nf} \nu(\alpha, \beta) = \omega^\alpha + \beta$. $\beta$ is not a successor, $\alpha$ is not 0.

(1.1) If $\gamma = \omega^{\delta+1}$, then $\gamma[i] = \begin{cases} \omega^\delta\cdot(i+1) & \text{if } \delta \ge \varepsilon_0, \\ \omega^\delta\cdot i & \text{if } \delta < \varepsilon_0. \end{cases}$

(1.2) If $\gamma = \omega^\alpha$, $\alpha$ is limit, then $\gamma[i] = \omega^{\alpha[i]}$.

(1.3) If $\gamma = \omega^\alpha + \beta$, $\beta$ is limit, then $\gamma[i] = \omega^\alpha + \beta[i]$.

(2) $\gamma =_{nf} \kappa(\alpha,\beta)$.

(2.1) If $\gamma = \kappa(\alpha,\beta)$, $\beta$ is limit, then $\gamma[i] = \kappa(\alpha,\beta[i])$.

(2.2) If $\gamma = \kappa(0,0) = \varepsilon_0$, then $\gamma[0] = \omega$, $\gamma[i+1] = \omega^{\gamma[i]}$.

(2.3) If $\gamma = \kappa(0,\eta+1) = \varepsilon_{\eta+1}$, then $\gamma[0] = \varepsilon_\eta+1$, $\gamma[i+1] = \omega^{\gamma[i]}$.

(2.4) If $\gamma = \kappa(\delta+1,0)$, then $\gamma[0] = \kappa(\delta,0)$, $\gamma[i+1] = \kappa(\delta,\gamma[i])$.

(2.5) If $\gamma = \kappa(\delta+1,\eta+1)$, then

$$\gamma[0] = \kappa(\delta+1,\eta)+1,\ \gamma[i+1] = \kappa(\delta,\gamma[i]).$$

(2.6) If $\gamma = \kappa(\alpha,0)$, $\alpha$ is limit, then $\gamma[i] = \kappa(\alpha[i],0)$.

(2.7) If $\gamma = \kappa(\alpha,\eta+1)$, $\alpha$ is limit, then $\gamma[i] = \kappa(\alpha[i],\kappa(\alpha,\eta)+1)$.

<u>Theorem 3.1.</u> P is a (1)-built-up system of fundamental sequences for $\Gamma_0$.

<u>Lemma 3.2.</u> Let $m < \omega$ and $\alpha$, $\beta$, $\gamma$, $\sigma$, $\tau < \Gamma_0$.

(1) If $\omega^\alpha + \beta > \beta$ and $\beta \xrightarrow[1]{} \gamma$, then $\omega^\alpha\cdot m + \beta \xrightarrow[1]{} \omega^\alpha\cdot m + \gamma$.

(2) For each $\alpha$ such that $1 < \alpha$, $\alpha \xrightarrow[1]{} 1$.

(3) If $\alpha \xrightarrow[1]{} \beta$ and there is no $\gamma$ such that $\alpha \overset{}{\underset{1}{\Longrightarrow}} \varepsilon_\gamma \xrightarrow[1]{} \beta$, then $\omega^\alpha \xrightarrow[1]{} \omega^\beta$.

(4) If $\beta \xrightarrow[1]{} \gamma$ and there is no $\delta$ such that $\beta \underset{1}{\Longrightarrow} \kappa(\alpha+1,\delta) \xrightarrow[1]{} \gamma$, then $\kappa(\alpha,\beta) \xrightarrow[1]{} \kappa(\alpha,\gamma)$.

(5) If $\alpha \xrightarrow[1]{} \beta$, then $\kappa(\alpha,0) \xrightarrow[1]{} \kappa(\beta,0)$.

(6) If $\alpha \xrightarrow[1]{} \beta$, $\alpha < \sigma$ and $\gamma = \kappa(\sigma,\tau)$ for some $\tau$, then $\kappa(\alpha,\gamma+1) \xrightarrow[1]{} \kappa(\beta,\gamma+1)$.

(7) Let $\alpha$ be limit, $\alpha =_{nf} \rho(\sigma,\tau)$, $\eta < \Gamma_0$. Then either the following condition (a) or (b) holds.

(a) there is an m such that all of the $\alpha[i]$ for $m < i$ are in the range of $\lambda\xi.\kappa(\eta,\xi)$.

(b) there is a $\xi < \Gamma_0$ such that $\kappa(\eta,\xi) \le \alpha[i] < \kappa(\eta,\xi+1)$ for all i or $\alpha[i] < \kappa(\eta,0)$ for all $i < \omega$.

(Proof) (1) For m = 1, by induction on $\beta$. For m > 1, by induction on m. (2),(3) By induction on $\alpha$. (4) By induction on $\alpha$ with subsidiary induction on $\beta$. (5),(6) By induction on $\alpha$. (7) By induction on $\alpha$; if $\rho = \nu$ then (b) holds.

(Proof of Theorem 3.1) By induction on $\gamma$, we will show that $\gamma[i+1] \xrightarrow[1]{} \gamma[i]$.

(1.1) $\gamma=\omega^{\delta+1}$. By Lemma 3.2(2), $\gamma[i+1]=\gamma[i]+\omega^{\delta} \overset{}{\underset{1}{\Longrightarrow}} \gamma[i]+1 \xrightarrow[1]{} \gamma[i]$.

(1.2) $\gamma=\omega^{\alpha}$, $\alpha$ is limit. By Lemma 3.2(3)(7) and ind.hyp.,

$$\gamma[i+1] = \omega^{\alpha[i+1]} \xrightarrow[1]{} \omega^{\alpha[i]} = \gamma[i].$$

(1.3) $\gamma=\omega^{\alpha}+\beta$, $\beta$ is limit. By Lemma 3.2(1) and ind.hyp.,

$$\gamma[i+1] = \omega^{\alpha}+\beta[i+1] \xrightarrow[1]{} \omega^{\alpha}+\beta[i] = \gamma[i].$$

(2.1) $\gamma=\kappa(\alpha,\beta)$, $\beta$ is limit. $\gamma[i]=\kappa(\alpha,\beta[i])$. Now (a) of Lemma 3.2(7) cannot hold for $\beta =_{nf} \kappa(\alpha+1,\tau)$. ( For then $\gamma = \lim_{i<\omega} \gamma[i] = \lim_{i<\omega} \kappa(\alpha,\beta[i]) = \lim_{i<\omega} \beta[i] = \beta$, it is contradiction. ) Hence (b) must hold. Therefore by Lemma 3.2(4) and ind.hyp.,

$$\gamma[i+1] = \kappa(\alpha,\beta[i+1]) \xrightarrow[1]{} \kappa(\alpha,\beta[i]) = \gamma[i].$$

(2.2) $\gamma=\varepsilon_0$. By Lemma 3.2(3) and induction on i,

$$\gamma[1] = \omega^{\omega} \xrightarrow[1]{} \omega = \gamma[0],\ \gamma[i+2] = \omega^{\gamma[i+1]} \xrightarrow[1]{} \omega^{\gamma[i]} = \gamma[i+1].$$

(2.3) $\gamma=\varepsilon_{\eta+1}$. By Lemma 3.2(2), $\gamma[1] = \omega^{\varepsilon_{\eta}+1} \xrightarrow[1]{} \omega^{\varepsilon_{\eta}}\cdot 2 \xrightarrow[1]{} \varepsilon_{\eta}+1 = \gamma[0]$. By Lemma 3.2(3) and induction on i,

$$\gamma[i+2] = \omega^{\gamma[i+1]} \xrightarrow[1]{} \omega^{\gamma[i]} = \gamma[i+1].$$

(2.4) $\gamma=\kappa(\delta+1,0)$. $\gamma[0]=\kappa(\delta,0)$, $\gamma[i+1]=\kappa(\delta,\gamma[i])$. Now $\gamma[i] <$

$\kappa(\delta+1,0)$. By Lemma 3.2(4) and induction on i,

$$\gamma[1] = \kappa(\delta,\gamma[0]) \xrightarrow[1]{} \kappa(\delta,0) = \gamma[0],$$

$$\gamma[i+2] = \kappa(\delta,\gamma[i+1]) \xrightarrow[1]{} \kappa(\delta,\gamma[i]) = \gamma[i+1].$$

(2.5) $\gamma=\kappa(\delta+1,\eta+1)$. By Lemma 3.2(6), $\gamma[1] = \kappa(\delta,\kappa(\delta+1,\eta)+1) \overset{}{\underset{1}{\Longrightarrow}}$ $\kappa(0,\kappa(\delta+1,\eta)+1) \xrightarrow[1]{} \omega^{\kappa(0,\kappa(\delta+1,\eta))+1} = \omega^{\kappa(\delta+1,\eta)+1} \xrightarrow[1]{}$ $\omega^{\kappa(\delta+1,\eta)}\cdot 2 \xrightarrow[1]{} \kappa(\delta+1,\eta)+1 = \gamma[0]$. Now $\kappa(\delta+1,\eta) < \gamma[i] <$ $\kappa(\delta+1,\eta+1)$. By Lemma 3.2(4) and induction on i,

$$\gamma[i+2] = \kappa(\delta,\gamma[i+1]) \xrightarrow[1]{} \kappa(\delta,\gamma[i]) = \gamma[i+1].$$

(2.6) $\gamma=\kappa(\alpha,0)$, $\alpha$ is limit. By Lemma 3.2(5),

$$\gamma[i+1] = \kappa(\alpha[i+1],0) \xrightarrow[1]{} \kappa(\alpha[i],0) = \gamma[i].$$

(2.7) $\gamma=\kappa(\alpha,\eta+1)$, $\alpha$ is limit. By Lemma 3.2(6) and ind.hyp.,

$$\gamma[i+1] = \kappa(\alpha[i+1],\kappa(\alpha,\eta)+1) \xrightarrow[1]{} \kappa(\alpha[i],\kappa(\alpha,\eta)+1) = \gamma[i].$$

## <u>§3. (n)-built-upness and Bachmann's property $\mathbb{B}[n]$</u>

In this section, we extend the theorems in Schmidt[4] by using (n)-built-up system. In the following, we assume that P is a system of fundamental sequences for $\Delta$.

<u>Definition 4.1.</u> P has property $\mathbb{B}[n]$ iff if $\alpha[i] < \mu \le \alpha[i+1]$ then $\alpha[i] \le \mu[n]$ for each limit $\alpha \in \Delta$, $i < \omega$ and $\mu \in \Delta$.

<u>Theorem 4.1.</u> P has property $\mathbb{B}[n]$ iff P is (n)-built-up.

(Proof) Let P have $\mathbb{B}[n]$, $\alpha \in \Delta$ be limit. Then for $\alpha[i] < \mu \le$ $\alpha[i+1]$, $\mu \xrightarrow[n]{} \alpha[i]$ by ind. on $\mu$, in particular, $\alpha[i+1] \xrightarrow[n]{} \alpha[i]$. Let P be (n)-built-up, $\alpha \in \Delta$ be limit and $\alpha[i] < \mu \le \alpha[i+1]$. We define $\langle\alpha_k\rangle_{k<\omega}$ inductively as follows: $\alpha_0 = \alpha[i+1]$,

$$\alpha_{k+1} = \begin{cases} \mu & \text{if } \alpha_k = \mu, \\ (\alpha_k)[j] & \text{if } \alpha_k \text{ is limit } > \mu, \text{ where } j \text{ is the least} \\ & \quad \text{such that } (\alpha_k)[j] > \mu, \\ \beta & \text{if } \alpha_k = \beta + 1. \end{cases}$$

Because P is (n)-built-up, we can prove $\alpha \geq \mu$ and $\alpha_k \xrightarrow[n]{} \alpha[i]$ by ind. on k. Then $\langle\alpha_k\rangle_{k<\omega}$ is non-ascending sequence, hence there is an $m < \omega$ such that $\alpha_k = \mu$ for $k \geq m$. Hence $\mu \xrightarrow[n]{} \alpha[i]$, in particular, $\alpha[i] \leq \mu[n]$.

Theorem 4.2. For any $n < \omega$, there is no assignment of fundamental sequences with property $\mathbb{B}[n]$ for the whole of the second number class.

(Proof) Let $\Omega$ be the first uncountable ordinal. Assume there is such a system for fixed $n < \omega$. Then $\lambda\alpha.\alpha[n]$ is a regressive function ( i.e. $\alpha[n] < \alpha$ for $0 < \alpha < \Omega$ ). and therefore there is an $A \subset \Omega$ of order type $\Omega$ and a $\beta \in \Omega$ such that $\alpha[n] = \beta$ for all $\alpha \in A$ (cf. Levy[2]). Let $\{\alpha_i : i < \omega\}$ be the first $\omega$ elements of A, $\alpha = \lim_{n<\omega} \alpha_i$ and $\langle\alpha[i]\rangle_{i<\omega}$ be the fundamental sequence for $\alpha$. Since $\alpha > \beta$, there is an $i < \omega$ such that $\beta < \alpha[i] < \alpha$. Let $m < \omega$ be the number such that $\alpha[i] < \alpha_m < \alpha$, and p be the greatest number such that $\alpha[p] < \alpha_m$. Then $\alpha[p] < \alpha_m \leq \alpha[p+1]$, so $\beta < \alpha[p] \leq \alpha_m[n]$. But $\alpha_m[n] = \beta$ because $\alpha_m \in A$. Contradiction.

Finally, we define properties $\mathbb{A}$ and $\mathbb{C}[n]$ to extend Remark of [4] as follows:

P has property $\mathbb{A}$ iff if $\beta < \alpha < \Delta$, then $\alpha \xrightarrow[n]{} \beta$ for some $n < \omega$.

P has property $\mathbb{C}[n]$ iff if $\langle F_\alpha \rangle_{\alpha<\Delta}$ is a sequence of number theoretic functions satisfies the properties $(a)_n$, $(b)_n$ and (c), and $\beta < \gamma \in \Delta$, then $F_\beta$ is dominated by $F_\gamma$ ( i.e. there is an m such that for all $n \geq m$, $F_\beta(n) < F_\gamma(n)$ ).

We can prove the following theorem which is an extension of Remark of [4].

<u>Theorem 4.3.</u> If P is (n)-built-up for some $n < \omega$, then P has properties $\mathbb{A}$ and $\mathbb{C}[n]$.

(Proof) Assume P is (n)-built-up. First we prove that P has $\mathbb{A}$ by ind. on $\alpha$. <u>Case 1.</u> $\alpha = 0$. Trivial. <u>Case 2.</u> $\alpha = \gamma + 1$. $\gamma \overset{}{\underset{m}{\Longrightarrow}} \beta$ for some $m < \omega$ by ind. hyp. Then $\alpha \xrightarrow[m]{} \gamma \underset{m}{\Longrightarrow} \beta$. <u>Case 3.</u> $\alpha$ is limit. $\alpha[i] > \beta$ for some i. Then $\alpha[i] \xrightarrow[k]{} \beta$ for some k by ind. hyp. Let $r = \max(n,k,i)$. By using Proposition 1.1, $\alpha \xrightarrow[r]{} \alpha[r] \underset{r}{\Longrightarrow} \alpha[i] \xrightarrow[r]{} \beta$. Next we prove that P has $\mathbb{C}[n]$. Because P has $\mathbb{A}$, $\gamma \xrightarrow[r]{} \beta$ for some r. If we put $m = \max(n,r)+1$, then by Theorem 1.3, $F_\beta(x) < F_\alpha(x)$ for $m \leq x < \omega$.

<u>References.</u>

[1] J.Ketonen and R.Solovay, Rapidly growing Ramsey functions, Ann. of Math. 113(1981), 267-314.

[2] A.Levy, Basic Set Theory, Springer-Verlag, 1979.

[3] D.Schmidt, Built-up systems of fundamental sequences and hierarchies of number theoretic functions, Arch. math. Logik

18(1976),47-53.

[4] D.Schmidt, Postscript to "Built-up systems of fundamental sequences and hierarchies of number theoretic functions", ibid. 18(1977),145-146.

[5] K.Schütte, Proof Theory, Springer-Verlag, 1977.

Noriya Kadota
Department of Applied Mathematics
Faculty of Engineering
Hiroshima University
Higashi-Hiroshima 724, Japan

Kiwamu Aoyama
Department of Mathematics
Faculty of Science
Kyushu University
Hukuoka 812, Japan