数理解析研究所講究録
第 661 巻 1988 年 231-241

A COMMENT ON THE DISTANT INTERACTION ALGORITHM

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I. INTRODUCTION

Recently Kraichnan[1,2] has presented the 'Distant Interaction Algorithm' (DSTA) in order to examine a turbulence theory of Yakhot and Orszag[3] which is based on the dynamic renormarization group and an extrapolation of the interactions among widely separated wavenumbers to the local ones. He has shown that the essence of the theory of Yakoht and Orszag can also be interpreted by DSTA of the perturbative closure and that the DSTA yields good inertial range constants such as the Kolmogorov constant $K_o = 1.56$ and the Obukhov Corrsin constant $C_o = 0.958$[1,2] as well as $K_o = 1.62$ and $C_o = 1.16$ of Yakhot and Orszag.[3] If both the DSTA and the theory of Yakhot and Orszag are reliable, they would provide the great advantages to the computation of the turbulence because they are very simple approximations and the eddy viscosity like expressions can be supported. On the other hand it has been believed that the local interactions among the wavenumbers are very important for the energy transfer in the inertial range.

In this paper we shall consider why the DSTA can provide good inertial range constants and discuss its reliability on the basis of a systematic La-

grangian closure theory.

II. THE DISTANT-INTERACTION ALGORITHM

We consider the evolution of the energy specrum $E(k)$ of the turbulence which obeys the incompressible Navier Stokes equation. The equation for the energy spectrum $E(k)$ can be written as

$$\left(\frac{\partial}{\partial t} + 2\nu_m k^2\right) E(k,t) = T(k,t), \tag{1}$$

where $T(k,t)$ is the energy transfer function due to the nonlinear term in the Navier Stokes equation. The requirement of energy conservation by the nonlinear interaction implies the following relation

$$\Pi(p) \equiv -\int_0^p dq\ T(q) = 2\int_0^p \nu(k \mid p) k^2 E(k) dk, \tag{2}$$

$$-2\nu(k \mid p) k^2 E(k) \equiv T(k \mid P), \tag{3}$$

where $T(k \mid p)$ is the contributions to the energy transfer function arising from the all triad wavenumbers interactions among $k < p$ and/or q. The relations (1) to (3) are exact.

In the DSTA we assume

$$\nu(k,t) = \nu(k \mid \beta k, t), \tag{4}$$

$$\nu(k \mid p,t) = \nu(0 \mid p,t), \tag{5}$$

where $\beta \geq 1$ is a cut off parameter. In any perturbative two point closures such as DIA[7], ALHDIA[8], SBALHDIA[9], TFM[10], EDQNM[11] and LRA,[5,6] $\nu(k \mid p)$ is given by the form

$$\nu(k \mid p) = \frac{1}{15}\int_p^\infty dq\ \theta(q,t)\left(5E(q) + q\frac{\partial}{\partial q}E(q)\right), \quad \text{for } k \ll p, \tag{6}$$

and

$$\theta(q,t) = \int_0^\infty ds\ \{G(q,t,s)\}^2, \tag{7}$$

where θ (G) is the so-called triple relaxation time (response function).

In order to specify G in (7), Kraichnan[1,2] assumed that the dynamics of the *Eulerian* velocity amplitude of the incompressible Navier-Stokes equation can be modeled by the Langevin equation

$$\left(\frac{\partial}{\partial t} + \nu_m k^2 + k^2 \nu(k,t)\right) u_i(\mathbf{k},t) = f_i(\mathbf{k},t), \tag{8}$$

where ν_m is molecular viscosity, $\nu(k,t)$ is a dynamical viscosity, and $f_i(\mathbf{k},t)$ the solenoidal forcing which has isortopy and fluctuates in time like white noise. As regards the interpretation of these quantities readers may refer Refs. 1 and 2. It follows from (8) that the response function $G(k,t,s)$ is given by

$$G(k,t,s) = \exp\left(-\int_s^t ds'\ (\nu_m k^2 + k^2 \nu(k,s'))\right), \tag{9}$$

and that the fluctuation dissipation relation holds as

$$Q(k,t,s) = G(k,t,s) Q(k,t,t), \quad t \geq s, \tag{10}$$

where

$$P_{ij}(\mathbf{k}) Q(k,t;s) = < u_i(\mathbf{k},t) u_j(-\mathbf{k},s) > . \tag{11}$$

The DSTA by Kraichnan is defined by (2), (4), (5), (6), (7) and (9). It is to be noted that the damping of G (and also θ) is essentially determined by the "viscosity" $\nu(k,t)$ or $\nu(k \mid p)$ appearing in (4) to (6) as implied by (9).

However this is not true in any systematic perturbative closures such as DIA and Lagrangian closures including the ALHDIA, SBALHDIA and LRA. We therefore reconsider the validity of the DSTA in the framework of such

a systematic perturbative closure. Here we use the Lagrangian renormalized approximation(LRA), because the LRA is the simplest among the systematic closures compatible with the the Kolmogorov inertial range law $k^{-5/3}$ and has been found in good agreement with the numerical and the field experiments.[6]

In the LRA we have

$$\left(\frac{\partial}{\partial t}+\nu_m k^2+k^2\eta(k,t,s)\right)v_i(\mathbf{k},t;s)=0,\quad t\geq s \tag{12}$$

$$\left(\frac{\partial}{\partial t}+\nu_m k^2+k^2\eta(k,t,s)\right)G_{ij}(\mathbf{k},t;s)=0,\quad t\geq s \tag{13a}$$

$$G(k,t;t)=\delta_{ij}, \tag{13b}$$

instead of (8) and (9), where $v_i(\mathbf{k},t;s)$ is the Fourier transform of the generalized velocity field $v_i(\mathbf{x},t;s)$ whose values are measured at time t of the fluid particle whose trajectory passes the x at time $s\leq t$, and the $G(k,t,s)$ is the response to the infinitesimal disturbance. The $\eta(k,t,s)$ is a eddy damping factor which comes from the lagrangian acceleration of the pressure[5,6];

$$\eta(k,t,s)=k\int_0^\infty dq\; q^3J(\frac{q}{k})\int_s^t ds\; G(q,t,s), \tag{14}$$

where

$$\begin{aligned} J(x)&=\pi\left((a^2-1)^2\log\frac{1+a}{|1-a|}-2a+\frac{10}{3}a^3\right)(2a^4)^{-1},\\ &\sim\frac{16\pi}{15}, \qquad\qquad \text{for}\quad x\ll 1, \end{aligned} \tag{15}$$

$$a=2x/(1+x^2).$$

It follows from eqs.(12) and (13) that the fluctuation dissipation relation holds

$$Q(k,t;s)=G(k,t,s)Q(k,s,s),\quad t\geq s, \tag{16}$$

where

$$P_{ij}(\mathbf{k})Q(k,t;s) = P_{il}(\mathbf{k}) < v_l(\mathbf{k},t;s)v_j(-\mathbf{k},s;s) >, \tag{17}$$

(cf. (11)) and from the white noise assumption we can write as

$$Q(k,t,s) = G(k,t,s)Q(k,t,t), \quad t \geq s. \tag{18}$$

The DSTA by the LRA is defined by (2), (4), (5), (6), (7) and (13).

III. ENERGY TRANSFER RATE IN THE INERTIAL RANGE

In the inertial range of steady turbulence the energy transfer rate $\Pi(p)$ is a constant

$$\Pi(p) = \epsilon, \tag{16}$$

where ϵ is a total rate of energy transfer. The DSTA and (16) yield the β dependent Kolmogorov constant $K(\beta)$ in the steady state inertial range spectrum

$$E(k) = K(\beta)\epsilon^{2/3}k^{-5/3}. \tag{17}$$

Kraichnan[1,2] assumed that there is a gap between k and βk in the energy spectrum and that the built up time of the triple correlation can be expressed by the dynamic viscosity as

$$\theta(k) = \frac{1}{2k^2\nu(k \mid \beta k)}, \tag{18}$$

and then obtained the approximate energy transfer rate

$$\tilde{\Pi}(p) = (\frac{21}{80})^{1/2}\{K(\beta)\}^{3/2}\beta^{-2/3}(1 + \frac{4}{3}\ln\beta)\epsilon, \tag{19}$$

$$\nu(k \mid p) = A(\beta)\epsilon^{1/3}p_\beta^{-4/3}, \quad (p_\beta = \max(p,\beta k), \quad p \geq k), \tag{20}$$

and

$$\theta(k) = \frac{1}{2A(\beta)}\beta^{4/3}\epsilon^{-1/3}k^{-2/3}, \tag{21}$$

where

$$A(\beta) = \left(\frac{7}{60}K(\beta)\right)^{1/2}\beta^{2/3}. \tag{22}$$

If the result of the perturbative closure (here LRA, but its choice is not important) $K_o = 1.72$[5] is used instead of $K(\beta)$, then we can see the performance of the approximation of the DSTA to the exact result ($\Pi(p) = \epsilon$) of the closure theory with respect to the energy transfer rate in the inertial range as

$$\tilde{\Pi}(p) = 1.16\epsilon, \quad \text{for} \quad \beta = 1, \tag{23}$$

$$\tilde{\Pi}(p) = 1.09\epsilon, \quad \text{for} \quad \beta = 8, \tag{24}$$

and the approximate Kolmogorov constant based on K_o can be estimated as follows

$$K = 1.56, \quad \text{for} \quad \beta = 1, \tag{25}$$

$$K = 1.62, \quad \text{for} \quad \beta = 8. \tag{26}$$

It seems, however, to be natural to consider that there is not the spectral gap in the inertial range and there is no reason for $\theta(k,t)$ to be expressed by the dynamic viscosity. Within the perturbative closure theory θ must be the time integral of the response function. If we use the exact results of the LRA[5](no extrapolation in the response eq.(3)), then we obtain

$$\theta_{LRA}(k) = \epsilon^{-1/3}k^{-2/3}\gamma\, I_2', \quad \gamma = \sqrt{\frac{2\pi}{K}}, \tag{27}$$

$$\begin{aligned} I_2' &= \int_0^\infty d\tau\, \overline{G}(\tau) \\ &= 1.49. \end{aligned} \tag{28}$$

Thus the dynamic viscosity and the approximate energy transfer rate become

$$\nu(k \mid p) = \frac{\sqrt{2\pi}}{6}\{K(\beta)\}^{1/2}\epsilon^{1/3} I_2' \, p_\beta^{-4/3}, \tag{29}$$

$$\tilde{\Pi}(p) = \frac{\sqrt{2\pi}}{4}\{K(\beta)\}^{3/2} I_2' \, \beta^{-4/3}(1 + \frac{4}{3}\ln\beta)\epsilon. \tag{30}$$

If we use the value K_o instead of $K(\beta)$, we obtain the following estimates

$$\tilde{\Pi}(p) = 0.58\epsilon, \quad K = 2.47, \quad \text{for} \quad \beta = 1, \tag{31}$$

$$\tilde{\Pi}(p) = 0.14\epsilon, \quad K = 6.38, \quad \text{for} \quad \beta = 8. \tag{32}$$

More consistent treatment of θ in the context of the DSTA may be an extrapolation of the response function $G(k,t,s)$. Under the same assumption which lead to (27), we estimate $\eta(k,t,s)$. By using (4) and the asymptotic expression of (5), we obtain

$$\eta(k \mid p) = B(\beta)\epsilon^{1/3} p_\beta^{-4/3} k^2, \tag{33}$$

$$\theta_{LRA,DSTA}(k) = \frac{1}{2B(\beta)}\epsilon^{-1/3} k^{-2/3} \beta^{4/3}, \tag{34}$$

$$B(\beta) = \left(\frac{2}{5}K(\beta)\right)^{1/2} \beta^{2/3}, \tag{35}$$

and

$$\nu(k \mid p) = \frac{1}{12B(\beta)} K(\beta)\epsilon^{1/3}\beta^{4/3} p_\beta^{-4/3}, \tag{36}$$

$$\tilde{\Pi}(p) = \left(\frac{5}{128}\right)^{1/2} \{K(\beta)\}^{3/2}\beta^{-2/3}(1 + \frac{4}{3}\ln\beta)\epsilon. \tag{37}$$

If we use again the value K_o instead of $K(\beta)$, we obtain the following estimates

$$\tilde{\Pi}(p) = 0.45\epsilon, \quad K = 2.95, \quad \text{for} \quad \beta = 1, \tag{38}$$

$$\tilde{\Pi}(p) = 0.42\epsilon, \quad K = 3.06, \quad \text{for} \quad \beta = 8. \tag{39}$$

For other combinations of the assumptions with respect to the energy gap and the choice of the θ, the results are shown in the Table. From these observations it is found that the use of θ due to the response function instead of the one due to ν leads to poor approximation of the energy transfer rate $\tilde{\Pi}(p)$. This reason will be discussed in the next section.

IV. MECHANISM OF DSTA

We consider here why the DSTA yields good inertial range constants. The built up time of the triple correlations θ is symbolically written as

$$\theta(k) = \frac{1}{2\eta(k \mid p)}. \tag{40}$$

As seen in Sec.III, the various approximations for η can be ordered as follows

$$\eta_{LRA} > \eta_{LRA,DSTA} > \nu_{DSTA}k^2, \tag{41}$$

where ν_{DSTA} means (20). On the other hand the factor $\{5E(q) + q\partial/\partial qE(q)\}$ has less contribution to $\tilde{\Pi}(p)$ than the exact one, then the two elements of the approximations in the DSTA work in a way to compensate each error. Moreover when $\nu_{DSTA}k^2$ is used in θ the residual error (say $R(\beta)$) of the Taylor expansion of the energy transfer function $T(k \mid p)$ can be reduced in the expression of ν_{DSTA}, because the equation for ν_{DSTA} must be self-consistently solved. That is, ν_{DSTA} may be symbolically expressed as

$$\nu_{DSTA} = \left(\int dq \frac{1}{2k^2} \{5E(q) + q\frac{\partial}{\partial q}E(q)\} \right)^{1/2} (1 + \frac{1}{2}R(\beta)).$$

When the Kolmogorov constant is computed, the reduction of the error $R(\beta)$ becomes larger than that in the situation for ν_{DSTA}. The equation for $\tilde{\Pi}$

is symbolically written as

$$\epsilon = \tilde{\Pi} = K^{3/2} C(\beta) \epsilon (1 + \frac{1}{2} R(\beta)),$$

thus we have

$$K = C(\beta)^{-2/3} (1 - \frac{1}{3} R(\beta)), \tag{42}$$

where C is a constant. That is the Kolmogorov constant is given as the root of the cubic equation. From (42) it is found that the error to the Kolmogorov constant is only 33% even if R is unity.

In the LRA, the essence of the reason why the DSTA gives good numerical constants is found to be in the compensation of errors in both the triple relaxation time and the Taylor expansion of the energy transfer function, and to lie in solving the self-consistent equation with identifying the dynamic eddy viscosity with the eddy damping η. At present we do not know whether such compensation of errors occurs or not for quantities other than the $\Pi(p)$ in the $k^{-5/3}$ inertial range.

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TABLE

	μ_{LRA}		$\mu_{LRA,DSTA}$		$\nu_{DSTA}k^2$	
	$\beta=1$	$\beta=8$	$\beta=1$	$\beta=8$	$\beta=1$	$\beta=8$
$\tilde{\Pi}/\epsilon$	0.58	0.14	0.45	0.42	0.98	0.92
	0.81	0.19	0.62	0.59	1.16*	1.09*
K	2.47	6.38	2.95	3.06	1.74	1.82
	2.00	5.20	2.36	2.44	1.56*	1.62*

Table

Comparison of the energy transfer rate and the Kolmogorov constant by the various Distant Interaction Algorithms. The upper lines of each comparison term show the values for no energy spectral gap and the lower for the energy gap. The value of the Kolmogorov constant by the exact closure theory is for example $K_o = 1.72$(LRA). The DSTA of Kraichnan[1,2] corresponds to the values with *.