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ON STARLIKE, CONVEX AND CLOSE-TO-CONVEX FUNCTIONS OF COMPLEX ORDER

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I. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U} = \{z: |z| < 1\}$. A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be starlike of complex order b ($b \neq 0$, complex) if and only if $f(z)/z \neq 0$ ($z \in \mathcal{U}$) and

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0 \qquad (z \in \mathcal{U}). \tag{1.2}$$

We denote by $S_0^*(b)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of complex order b. A function $f(z)$ in $\mathcal{A}$ is said to be convex of complex order b ($b \neq 0$, complex) if and only if $f'(z) \neq 0$ ($z \in \mathcal{U}$) and

$$\mathrm{Re}\left\{1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z \in \mathcal{U}). \tag{1.3}$$

Also we denote by $K_0(b)$ the subclass of $\mathcal{A}$ consisting of functions which are convex of complex order b. Note that $f(z) \in K_0(b)$ if and only if $zf'(z) \in S_0^*(b)$.

REMARK 1. The class $S_0^*(b)$ was introduced by Nasr and Aouf [3], and the class $K_0(b)$ was introduced by Nasr and Aouf [4].

REMARK 2. Letting $b = 1 - \alpha$, we observe that $S_0^*(1-\alpha) = S^*(\alpha)$ and $K_0(1-\alpha) = K(\alpha)$, where $S^*(\alpha)$ and $K(\alpha)$ denote the classes of starlike functions and convex functions of order α, respectively.

A function f(z) belonging to the class $\mathcal{A}$ is said to be close-to-convex of complex order b ($b \neq 0$, complex) if and only if there exists a function $g(z) \in S_0^*(1)$ satisfying the condition

$$\text{Re}\left\{1 + \frac{1}{b}\left(\frac{f'(z)}{g'(z)} - 1\right)\right\} > 0 \qquad (z \in \mathcal{U}). \tag{1.4}$$

We denote by $C_0(b)$ the subclass of $\mathcal{A}$ consisting of functions which are close-to-convex of complex order b. We note that $C_0(1-\alpha) = C(\alpha)$, where $C(\alpha)$ is the class of close-to-convex functions of order α.

2. FORMER RESULTS

THEOREM A (Nasr and Aouf [3]). A function f(z) is in the class $S_0^*(b)$ if and only if there exists a probability measure $\mu(t)$ $(0 \leq t < 2\pi)$ such that

$$f(z) = z\exp\left\{\int_0^{2\pi} -2b\log(1 - ze^{-it})\,d\mu(t)\right\}. \tag{2.1}$$

THEOREM B (Nasr and Aouf [3]). If a function f(z) is in the class $S_0^*(b)$ with $\text{Re}(b) > 0$, then

$$|a_n| \leq \frac{1}{(n-1)!}\prod_{m=0}^{n-2}|2b + m| \qquad (n \geq 2). \tag{2.2}$$

The equality in (2.2) is attained for the function

$$f(z) = \frac{z}{(1-z)^{b}} = z + \sum_{n=2}^{\infty}\prod_{m=0}^{n-2}\left(\frac{2b+m}{m+1}\right)z^n. \tag{2.3}$$

THEOREM C (Nasr and Aouf [3]). If a function f(z) is in the class $S_0^*(b)$, then

$$|a_3 - \mu a_2^2| \leq |b|\max(1, |4b\mu - 2b - 1|), \tag{2.4}$$

where μ is a complex number.

THEOREM D (Nasr and Aouf [3]). If a function f(z) is in the class

$S_0^*(b)$, then

$$\left| \frac{zf'(z)}{f(z)} \right| \leqq 1 + \frac{2|b|r}{1-r} \qquad (|z| = r < 1). \tag{2.5}$$

The equality in (2.5) is attained for the function

$$f(z) = \frac{z}{(1-z)^{2b}}. \tag{2.6}$$

3. SOME RESULTS

In order to derive our results, we have to recall here the following lemma.

LEMMA I (Miller and Mocanu [2]). Let $q(z)$ be univalent in the unit disk $\mathbb{U}$, $\theta(w)$ and $\phi(w)$ be analytic in the domain $\mathbb{D}$ containing $q(\mathbb{U})$, and $\phi(w) \neq 0$ for $w \in q(\mathbb{U})$. Set

$$Q(z) = zq'(z)\phi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + Q(z).$$

Suppose that

(i) $Q(z)$ is starlike (univalent) in $\mathbb{U}$ with $Q(0) = 0$, $Q'(0) \neq 0$;

and

(ii) $$\operatorname{Re}\left\{ \frac{zh'(z)}{Q(z)} \right\} = \operatorname{Re}\left\{ \frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} \right\} > 0 \qquad (z \in \mathbb{U}).$$

If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = q(0)$, $p(\mathbb{U}) \subset \mathbb{D}$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \tag{3.1}$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of the subordination (3.1).

REMARK 3. The univalent function $q(z)$ is said to be a dominant of the differential subordination

$$\Psi(p(z), zp'(z)) \prec h(z) \tag{3.2}$$

if $p(z) \prec q(z)$ for all $p(z)$ satisfying (3.2). In particular, if $\tilde{q}(z)$ is a dominant of (3.2) and $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (3.2), then $\tilde{q}(z)$ is said to be the best dominant of (3.2).

THEOREM I (Obradović, Aouf and Owa [5]). If a function $f(z)$ is in the class $S_0^*(b)$, then

$$\left(\frac{f(z)}{z}\right)^a \prec \frac{1}{(1-z)^{2ab}}, \tag{3.3}$$

where a is a complex number, $a \neq 0$, and either $|2ab+1| \leqq 1$ or $|2ab-1| \leqq 1$. The function $1/(1-z)^{2ab}$ is the best dominant of the differential subordination (3.3).

PROOF. Letting $q(z) = 1/(1-z)^{2ab}$, $\theta(w) = 1$ and $\phi(w) = 1/abw$ in Lemma 1, we see that $Q(z) = 2z/(1-z)$ and $h(z) = (1+z)/(1-z)$. Therefore, $Q(z)$ is starlike (univalent) in the unit disk $\mathbb{U}$, $Q(0) = 0$, $Q'(0) = 2 \neq 0$, and

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\frac{1}{1-z}\right\} > 0 \qquad (z \in \mathbb{U}).$$

Thus, the conditions (i) and (ii) in Lemma 1 are satisfied.
Also, we see that $q(z)$ is univalent in $\mathbb{U}$ by Royster [6]. We define the function $p(z)$ by $p(z) = (f(z)/z)^a$ for $f(z) \in S_0^*(b)$. Then $p(z)$ is analytic in $\mathbb{U}$, $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, and $p(z) \neq 0$ for $0 < |z| < 1$. Since

$$\theta(p(z)) + zp'(z)\phi(p(z)) = 1 + \frac{1}{ab}\frac{zp'(z)}{p(z)} \tag{3.4}$$

$$= 1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right),$$

$f(z) \in S_0^*(b)$ implies that

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1+z}{1-z} = h(z). \tag{3.5}$$

Consequently, with the help of Lemma 1, we observe that

$$(3.6)\qquad 1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right) \prec \frac{1+z}{1-z}$$

$$\Longrightarrow \left(\frac{f(z)}{z}\right)^{a} \prec \frac{1}{(1-z)^{2ab}} .$$

This completes the assertion of Theorem 1.

Taking a = -1/2b in Theorem 1, we have

COROLLARY 1. If a function f(z) is in the class $S_0^*(b)$, then

$$(3.7)\qquad \left(\frac{z}{f(z)}\right)^{1/2b} \prec 1-z,$$

and

$$(3.8)\qquad \left|\left(\frac{z}{f(z)}\right)^{1/2b}-1\right| < |z| \qquad (z \in U).$$

REMARK 4. If $b = 1-\alpha$ $(0 \leqq \alpha < 1)$, Corollary 1 becomes that

$$f(z) \in S^*(\alpha) \Longrightarrow \left|\left(\frac{z}{f(z)}\right)^{1/2(1-\alpha)}-1\right| < |z| \qquad (z \in U).$$

This is the former result given by Todorov [8].

COROLLARY 2. If a function f(z) is in the class $K_0(b)$, then

$$(3.9)\qquad (f'(z))^{a} \prec \frac{1}{(1-z)^{2ab}} ,$$

where a is a complex number, $a \neq 0$, and either $|2ab+1| \leqq 1$ or $|2ab-1| \leqq 1$.

Next, we show

THEOREM 2 (Obradović, Aouf and Owa [5]). Let a function f(z) be in the class $S_0^*(b)$, and a be a complex number, $a \neq 0$, $0 < 2ab \leq 1$. Then

$$\text{(3.10)} \qquad \operatorname{Re}\left(\frac{f(z)}{z}\right)^a > \frac{1}{2^{2ab}} \qquad (z \in U)$$

and

$$\text{(3.11)} \qquad \left|\left(\frac{f(z)}{z}\right)^{-a} - 2^{2ab-1}\right| < 2^{2ab-1} \qquad (z \in U).$$

The estimates are best possible.

PROOF. Define the function $g(\theta) = \cos(\mu\theta) - \cos^{\mu}\theta$ for $0 < \mu \leq 1$ and $-\pi/2 \leq \theta \leq \pi/2$. Then we see that $g(\theta)$ is an even function of θ and

$$\begin{aligned} g'(\theta) &= \mu\{\cos^{\mu-1}\theta \sin\theta - \sin(\mu\theta)\} \\ &\geq \mu\{\sin\theta - \sin(\mu\theta)\} \\ &\geq 0 \end{aligned}$$

for $0 < \mu \leq 1$ and $0 \leq \theta \leq \pi/2$. This proves that $g(\theta) \geq g(0) = 0$, that is, that

$$\cos(\mu\theta) - \cos^{\mu}\theta \geq 0$$

for $0 < \mu \leq 1$ and $-\pi/2 \leq \theta \leq \pi/2$.

Letting $\mu = 2ab$ and $\theta = \phi/2 - \pi/2$ $(0 \leq \phi \leq 2\pi)$, we obtain that

$$\begin{aligned} \operatorname{Re}\left\{\frac{1}{(1 - e^{i\phi})^{2ab}}\right\} &= \left(2\sin\frac{\phi}{2}\right)^{-\mu} \cos\{\mu(\phi/2 - \pi/2)\} \\ &= (2\cos\theta)^{-\mu}\cos(\mu\theta). \end{aligned}$$

Therefore, using Theorem 1, we have

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^a > \operatorname{Re}\left\{\frac{1}{(1 - e^{i\phi})^{2ab}}\right\} \geq \frac{1}{2^{2ab}} \qquad (0 \leq \phi \leq 2\pi).$$

This completes the first half of the theorem.

The second half of the theorem follows from the above and the result given by Wilken and Feng [10].

Further, taking the function $f(z) = z/(1-z)^{2b}$, we see that the results of the theorem are sharp.

In order to derive our next results, we need the following lemma.

LEMMA 2 (Jack [1]). Let $w(z)$ be regular in the unit disk $\mathcal{U}$ and such that $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_0, then we have

$$z_0 w'(z_0) = k w(z_0),$$

where k is real and $k \geqq 1$.

With the aid of the above lemma, we prove

THEOREM 3. If a function $f(z)$ belonging to $\mathcal{A}$ satisfies

$$\left| \frac{f'(z)}{g'(z)} - 1 \right|^{\alpha} \left| \frac{zf''(z)}{g'(z)} - \frac{zf'(z)g''(z)}{g'(z)^2} \right|^{\beta} < |b|^{\alpha+\beta} \quad (z \in \mathcal{U}) \tag{3.12}$$

for some $\alpha \geqq 0$, $\beta \geqq 0$, and $g(z) \in S_0^*(1)$, then $f(z) \in \mathcal{C}_0(b)$.

PROOF. Defining the function $w(z)$ by

$$w(z) = \frac{1}{b}\left(\frac{f'(z)}{g'(z)} - 1 \right) \tag{3.13}$$

for $f(z)$ belonging to $\mathcal{A}$ and $g(z)$ belonging to $S_0^*(1)$, we see that $w(z)$ is regular in the unit disk $\mathcal{U}$ and $w(0) = 0$. Noting that

$$bzw'(z) = \frac{zf''(z)}{g'(z)} - \frac{zf'(z)g''(z)}{g'(z)^2}, \tag{3.14}$$

we know that (3.12) can be written

$$|bw(z)|^{\alpha} |bzw'(z)|^{\beta} < |b|^{\alpha+\beta}. \tag{3.15}$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1. \tag{3.16}$$

Then, Lemma 2 leads to

$$\begin{aligned} |bw(z_0)|^{\alpha} |bz_0 w'(z_0)|^{\beta} &= |b|^{\alpha+\beta} k^{\beta} \qquad (k \geqq 1) \\ &\geqq |b|^{\alpha+\beta} \end{aligned} \tag{3.17}$$

which contradicts our condition (3.12). Therefore, we conclude that $|w(z)| < 1$ for all $z \in U$, that is, that

$$\left| \frac{1}{b}\left(\frac{f'(z)}{g'(z)} - 1 \right) \right| < 1 \qquad (z \in U).$$

This implies that

$$\mathrm{Re}\left\{ 1 + \frac{1}{b}\left(\frac{f'(z)}{g'(z)} - 1 \right) \right\} > 0 \qquad (z \in U),$$

which proves that $f(z) \in C_0(b)$.

Letting $g(z) = z \in S_0^*(1)$ in Theorem 3, we have

COROLLARY 3. If a function $f(z)$ belonging to A satisfies

$$|f'(z) - 1|^{\alpha} |zf''(z)|^{\beta} < |b|^{\alpha+\beta} \qquad (z \in U) \tag{3.18}$$

for some $\alpha \geqq 0$, $\beta \geqq 0$, then $f(z) \in C_0(b)$.

THEOREM 4. If a function $f(z)$ belonging to A satisfies

$$\left| a\left(\frac{zf'(z)}{f(z)} - 1 \right) + (1 - a) \frac{z^2 f''(z)}{f(z)} \right| < |b| \qquad (z \in U) \tag{3.19}$$

for $0 \leqq a \leqq 1$, $|b| \leqq 1$, then $f(z) \in S_0^*(b)$.

PROOF. Let

$$w(z) = \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1 \right) \tag{3.20}$$

for $f(z) \in \mathcal{A}$. Then the function $w(z)$ is regular in the unit disk $\mathcal{U}$ and $w(0) = 0$. Making use of the logarithmic differentiations in both sides of (3.20), we have

$$\frac{zf''(z)}{f'(z)} = bw(z) + \frac{bzw'(z)}{1 + bw(z)} . \tag{3.21}$$

It follows that

$$a\left(\frac{zf'(z)}{f(z)} - 1 \right) + (1 - a)\frac{z^2 f''(z)}{f(z)} \tag{3.22}$$

$$= bw(z)\left\{1 + (1 - a)bw(z) + (1 - a)\frac{zw'(z)}{w(z)}\right\},$$

or

$$\left|bw(z)\left\{1 + (1 - a)\left(bw(z) + \frac{zw'(z)}{w(z)}\right)\right\}\right| < |b|. \tag{3.23}$$

Assume that there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then we can write $w(z_0) = e^{i\theta}$. Therefore, using Lemma 2, we see that

$$\left|bw(z_0)\left\{1 + (1 - a)\left(bw(z_0) + \frac{z_0 w'(z_0)}{w(z_0)}\right)\right\}\right| \tag{3.24}$$

$$= |b||1 + (1 - a)(k + be^{i\theta})|$$

$$\geqq |b||1 + (1 - a)(1 - |b|)|$$

$$\geqq |b|$$

which contradicts (3.23). Thus we have

(3.25) $$|w(z)| = \left| \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right| < 1 \qquad (z \in \mathbb{U}).$$

Noting that (3.25) implies

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0 \qquad (z \in \mathbb{U}),$$

we complete the proof of Theorem 4.

Further, we derive

THEOREM 5. If a function $f(z)$ belonging to $\mathcal{A}$ satisfies

(3.26) $$\left| \frac{f'(z)}{g'(z)} - 1\right|^{\alpha} \left| \frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)} \right|^{\beta} < |b|^{\alpha}\left(\frac{|b|}{1 + |b|} \right)^{\beta}$$

for some $\alpha \geqq 0$, $\beta \geqq 0$, $g(z) \in S_0^*(1)$, and for all $z \in \mathbb{U}$, then $f(z) \in \mathcal{C}_0(b)$.

PROOF. Defining the function $w(z)$ by (3.13), we have

(3.27) $$\frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)} = \frac{bzw'(z)}{1 + bw(z)}.$$

Therefore, the condition (3.26) leads to

(3.28) $$|bw(z)|^{\alpha}\left| \frac{bzw'(z)}{1 + bw(z)} \right|^{\beta} < |b|^{\alpha}\left(\frac{|b|}{1 + |b|}\right)^{\beta}.$$

Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

(3.29) $$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then letting $w(z_0) = e^{i\theta}$, and applying Lemma 2, we obtain that

(3.30) $$|bw(z_0)|^{\alpha}\left| \frac{bz_0w'(z_0)}{1 + bw(z_0)} \right|^{\beta} = |b|^{\alpha}\left| \frac{bkw(z_0)}{1 + bw(z_0)} \right|^{\beta}$$
$$\geqq |b|^{\alpha}\left(\frac{|b|}{1 + |b|} \right)^{\beta}$$

which contradicts our condition (3.26). Thus we see that $f(z) \in C_0(b)$.

Taking $g(z) = z \in S_0^*(1)$ in Theorem 5, we have

COROLLARY 4. If a function $f(z)$ belonging to A satisfies

$$|f'(z) - 1|^{\alpha} \left| \frac{zf''(z)}{f'(z)} \right|^{\beta} < |b|^{\alpha}\left(\frac{|b|}{1+|b|}\right)^{\beta} \qquad (z \in U) \tag{3.31}$$

for some $\alpha \geqq 0$, $\beta \geqq 0$, then $f(z) \in C_0(b)$.

4. COEFFICIENT INEQUALITIES

In this section, we consider the coefficient inequalities of functions to be in the classes $S_0^*(b)$, $K_0(b)$ and $C_0(b)$.

THEOREM 6. If a function $f(z)$ belonging to A satisfies

$$\sum_{n=2}^{\infty} (n + |b| - 1)|a_n| \leqq |b| \qquad (b \text{ is complex, } b \neq 0), \tag{4.1}$$

then $f(z) \in S_0^*(b)$.

PROOF. It is easy to see that the condition (4.1) implies that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{\sum_{n=2}^{\infty} na_n z^{n-1}}{1 - \sum_{n=2}^{\infty} a_n z^{n-1}} \right| < \frac{\sum_{n=2}^{\infty} n|a_n|}{1 - \sum_{n=2}^{\infty} |a_n|} \leqq |b|. \tag{4.2}$$

Thus we have $f(z) \in S_0^*(b)$.

Similarly, we have

THEOREM 7. If a function $f(z)$ belonging to $\mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty} n(n+|b|-1)|a_n| \leqq |b| \qquad (b \text{ is complex, } b \neq 0), \tag{4.3}$$

then $f(z) \in \mathcal{K}_0(b)$.

REMARK 5. Letting $b = 1 - \alpha$ $(0 \leqq \alpha < 1)$ in Theorem 6 and Theorem 7, we have the corresponding results by Silverman [7].

THEOREM 8. If a function $f(z)$ belonging to $\mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty} n|a_n| \leqq |b| \qquad (b \text{ is complex, } b \neq 0), \tag{4.4}$$

then $f(z) \in \mathcal{C}_0(b)$.

Now, we introduce the subclasses $\mathcal{S}_1^*(b)$, $\mathcal{K}_1(b)$ and $\mathcal{C}_1(b)$ of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the coefficient inequalities (4.1), (4.3) and (4.4), respectively.

REMARK 6. It follows from the definitions of $\mathcal{S}_1^*(b)$, $\mathcal{K}_1(b)$ and $\mathcal{C}_1(b)$ that

$$\text{(i) } f(z) \in \mathcal{S}_1^*(b) \Longrightarrow |a_n| \leqq \frac{|b|}{n+|b|-1} \qquad (n \geqq 2),$$

$$\text{(ii) } f(z) \in \mathcal{K}_1(b) \Longrightarrow |a_n| \leqq \frac{|b|}{n(n+|b|-1)} \qquad (n \geqq 2),$$

$$\text{(iii) } f(z) \in \mathcal{C}_1(b) \Longrightarrow |a_n| \leqq \frac{|b|}{n} \qquad (n \geqq 2).$$

Next, we prove

THEOREM 9. If a function f(z) is in the class $S_1^*(b)$, then

(4.5) $$|z| - \frac{|b|}{1+|b|}|z|^2 \leqq |f(z)| \leqq |z| + \frac{|b|}{1+|b|}|z|^2$$

for $z \in \mathcal{U}$. If $0 < |b| \leqq 1$, then

(4.6) $$1 - \frac{2|b|}{1+|b|}|z| \leqq |f'(z)| \leqq 1 + \frac{2|b|}{1+|b|}|z|$$

for $z \in \mathcal{U}$. The estimates in (4.5) and (4.6) are sharp.

PROOF. Noting that

(4.7) $$\sum_{n=2}^{\infty}|a_n| \leqq \frac{|b|}{1+|b|}$$

for $f(z) \in S_1^*(b)$, we have

(4.8) $$|f(z)| \geqq |z| - |z|^2 \sum_{n=2}^{\infty}|a_n| \geqq |z| - \frac{|b|}{1+|b|}|z|^2$$

and

(4.9) $$|f(z)| \leqq |z| + |z|^2 \sum_{n=2}^{\infty}|a_n| \leqq |z| + \frac{|b|}{1+|b|}|z|^2.$$

Further, if $0 < |b| \leqq 1$, then we have

(4.10) $$\frac{1+|b|}{2}\sum_{n=2}^{\infty}n|a_n| \leqq \sum_{n=2}^{\infty}(n+|b|-1)|a_n| \leqq |b|,$$

or

(4.11) $$\sum_{n=2}^{\infty}n|a_n| \leqq \frac{2|b|}{1+|b|}.$$

Thus we obtain

(4.12) $$|f'(z)| \geqq 1 - |z| \sum_{n=2}^{\infty}n|a_n| \geqq 1 - \frac{2|b|}{1+|b|}|z|$$

and

$$(4.13) \qquad |f'(z)| \leqq 1 + |z| \sum_{n=2}^{\infty} n|a_n| \leqq 1 + \frac{2|b|}{1+|b|}|z|.$$

Finally, taking the function

$$(4.14) \qquad f(z) = z - \frac{|b|}{1+|b|} z^2 ,$$

we see that the estimates in (4.5) and (4.6) are sharp.

Using the same manner as in the proof of Theorem 9, we have

THEOREM 10. If a function f(z) is in the class $K_1(b)$, then

$$(4.15) \qquad |z| - \frac{|b|}{2(1+|b|)}|z|^2 \leqq |f(z)| \leqq |z| + \frac{|b|}{2(1+|b|)}|z|^2$$

and

$$(4.16) \qquad 1 - \frac{|b|}{1+|b|}|z| \leqq |f'(z)| \leqq 1 + \frac{|b|}{1+|b|}|z|$$

for $z \in U$. The estimates in (4.15) and (4.16) are sharp for the function

$$(4.17) \qquad f(z) = z - \frac{|b|}{2(1+|b|)} z^2.$$

THEOREM 11. If a function f(z) is in the class $C_1(b)$, then

$$(4.18) \qquad |z| - \frac{|b|}{2}|z|^2 \leqq |f(z)| \leqq |z| + \frac{|b|}{2}|z|^2$$

and

$$(4.19) \qquad 1 - |b||z| \leqq |f'(z)| \leqq 1 + |b||z|$$

for $z \in U$. The estimates in (4.8) and (4.9) are sharp for the function

$$(4.20) \qquad f(z) = z - \frac{|b|}{2} z^2.$$

Finally, we give

CONJECTURE. Let a function $f(z) \in \mathcal{A}$ be defined by

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0). \tag{4.21}$$

Then

(i) $f(z) \in S_0^*(b)$ if and only if

$$\sum_{n=2}^{\infty} (n + |b| - 1)a_n \leqq |b|.$$

(ii) $f(z) \in K_0(b)$ if and only if

$$\sum_{n=2}^{\infty} n(n + |b| - 1)a_n \leqq |b|.$$

(iii) $f(z) \in C_0(b)$ with $g(z) = z \in S_0^*(1)$ if and only if

$$\sum_{n=2}^{\infty} na_n \leqq |b|.$$

REMARK 7. The above conjectures (i), (ii) and (iii) are true for $b = 1 - \alpha$ $(0 \leqq \alpha < 1)$.

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