数理解析研究所講究録
第 679 巻 1989 年 117-126

On Higher Differentiability and Partial Regularity of the Minimizers in the Calculus of Variations

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1.Introduction

In this paper we shall treat with the following problem in the calculus of variations : Let n and N be positive integers and suppose that $\Omega \subset R^n$ is a bounded domain with the C^2 -class boundary. Then we consider the functional,

$$I[v] \equiv \sum_{\alpha,\beta=1}^{n} \sum_{i,j=1}^{N} \int_{\Omega} a_{i,j}^{\alpha,\beta}(x,v) D_\alpha v^i D_\beta v^j dx \qquad \text{for } v : \Omega \mapsto R^N , \tag{1.1}$$

where $D_\alpha v^i = \frac{\partial v^i}{\partial x_\alpha}$ $(\alpha = 1, \cdots, n \ , i = 1, \cdots, N)$ and $a_{i,j}^{\alpha,\beta}$ $(\alpha, \beta = 1, \cdots n \,, i, j = 1, \cdots, N)$ are continuously differentiable functions in $\Omega \times R^N$ satisfying the following : There exist positive numbers λ and Λ $(0 < \lambda \le \Lambda < +\infty)$ such that $a_{i,j}^{\alpha,\beta}$ $(\alpha, \beta = 1, \cdots, n \,, i, j = 1, \cdots, N)$ satisfy for $^\forall(x,v) \in \Omega \times R^N$

$$\lambda|\zeta|^2 \le \sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{n} a_{i,j}^{\alpha,\beta}(x\,,v)\zeta_\alpha^i \zeta_\beta^j \le \Lambda|\zeta|^2 \qquad \text{for } \ ^\forall(x,v) \in R^{n\times N} \text{ and } ^\forall\zeta \in R^{n\times N} , \tag{1.2}$$

$$a_{i,j}^{\alpha,\beta} = a_{j,i}^{\beta,\alpha} . \tag{1.3}$$

In addition , since the coefficients $a_{i,j}^{\alpha,\beta}$ belong to $C^1(\Omega \times R^N \,; R)$, for positive numbers K_1 and K_2 there exists a positive number $L(K_1, K_2)$ such that

$$\max_{\substack{1\le i,j\le N \\ 1\le\alpha,\beta\le n}} \max_{\substack{|x|\le K_1 \\ |z|\le K_2}} |a_{i,j}^{\alpha,\beta}(x,z)| + \max_{\substack{1\le i,j\le N \\ 1\le\alpha,\beta\le n}} \max_{\substack{|x|\le K_1 \\ |z|\le K_2}} |\frac{\partial a_{i,j}^{\alpha,\beta}}{\partial e}(x,z)| \tag{1.4}$$

$$+ \max_{\substack{1\le i,j,k\le N \\ 1\le\alpha,\beta\le n}} \max_{\substack{|x|\le K_1 \\ |z|\le K_2}} |a_{i,j,k}^{\alpha,\beta}(x,z)| \le L(K_1, K_2)$$

$$\text{where} \quad a_{i,j,k}^{\alpha,\beta}(x,z) \equiv \frac{\partial a_{i,j}^{\alpha,\beta}}{\partial z_k}(x,z)$$

and

$$\frac{\partial a_{i,j}^{\alpha,\beta}}{\partial e}(x,z) \text{denotes the derivative in a direction of a vector } e \text{ in } R^n .$$

This implies the existence of at least a minimizer of the functional I in the Sobolev space $H^{1,2}(\Omega\,;R^N)$ and I is lower semicontinuous with respect to the weak topology of $H^{1,2}(\Omega\,;R^N)$ (see [Mo]) under an appropriate boundary condition .

First , we show that the first-derivatives of minimizers satisfies a modulus of uniform continuity in the norm $L^2_{loc}(\Omega\,;R^N)$.

Secondly , we mention a convergence theorem and a partial regularity result of the weak differentials of minimizers. However , we remark that the former theorem was proved in [Gm] , [HKL] and [Mm] .

We use the summation convention that Latin indices run from *1* to N and Greek indices run from *1* to n .

We conclude this introduction by recalling other notational conventions :

(1.5) $$B_R(x_0) \equiv \{x \in R^N \,:\, |x - x_0| < R\}.$$

For a set $A \subset R^N$, we denote by $mesA$ and $|A|$ the n−dimensional Lebesgue measure of A .

For $u \in L^1(B_R(x_0)\,;R^N)$, we define

(1.6) $$u_{x_0,R} = \frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} u(x)dx .$$

For a sufficiently small number d , we define an open set

(1.7) $$\Omega_d = \Omega - \{x \in \Omega \,:\, dist(x,\partial\Omega) \le d\},$$

where $dist(x,\partial\Omega)$ means the Euclidean metric between x and $\partial\Omega$.

For a set A in R^n , $\mathrm{H}^{(k)}(A)$ denotes the k− dimensional Hausdorff measure of A (for the definition , see [Gm]) .

$e_i\,(i = 1,\cdots,n)$ means the unit vector in R^N parallel to the x_i− axis. We define a translate operator Δ_m $(m = 1,2,\cdots,n)$ by

(1.8) $$(\Delta_m f)(x) = f(x + he_m) - f(x) \qquad \text{for} \qquad f \in L^p(\Omega\,;R^N).$$

2. Main Result

Under the above preparations , we can describe

THEOREM 1. *Let u be a minimizer of the functional I in $H^{1,2}(\Omega\,;R^N)$ and let us suppose that u is a bounded , namely there exists some positive constant M such that ess.sup$|u| \le M$. Then , for any fixed domain $\tilde{\Omega}$ compactly contained in Ω , there exists positive number $\alpha = \alpha\,(n, N, \lambda, \Lambda, M, L)$ $(0 < \alpha \le 1)$ and $C = C(n, N, \lambda, \Lambda, \tilde{\Omega}, \Omega, M, L)$ such that for $h > 0$ with $h \le dist(\tilde{\Omega}\,;\partial\Omega)$ u satisfies*

$$\int_{\tilde{\Omega}} |\Delta_m(\nabla u(x))|^2 dx \le C\cdot h^\alpha \qquad \text{for} \qquad {}^\forall m\,(m = 1,2,\cdots,n)$$

THEOREM 2. *Suppose that* $\{u_i\}_{i\geq 1}$ *is a sequence of minimizers of* I *in the space* $H^{1,2}(\Omega; R^N)$ *such that* $\{u_i\}_{i\geq 1}$ *converges strongly to a function* u_0 *in* $L^2_{loc}(\Omega; R^N)$. *Then the function* u_0 *belongs to* $H^{1,2}_{loc}(\Omega; R^N)$ *and moreover a suitable subsequence of* $\{u_i\}_{i\geq 1}$ *converges strongly to* u_0 *in* $H^{1,2}_{loc}(\Omega; R^N)$.

THEOREM 3. *Let* u *be a minimizer of the functional* I *in* $H^{1,2}(\Omega; R^N)$. *Then , for a singular set defined by*

$$S = \{x \in \Omega : \nexists \lim_{\varrho\to +0} |(Du)_{x,\varrho}|\} \cup \{x \in \Omega : \lim_{\varrho\to +0} |(Du)_{x,\varrho}| = +\infty\} \tag{2.1}$$

the following

$$H^{(\beta)}(S) = 0 \tag{2.2}$$

holds for any positive number β *satisfying* $n - 2\alpha < \beta < n$.

Remark. In the following proof , the letter C_i ($i = 1, \cdots, 14$) means a various constant depending only on n, N, λ, Λ, Ω, $\tilde{\Omega}$, M and L.

PROOF OF THEOREM 1

First , a minimizer u is a weak solution of *the Euler-Lagrange* equations of the functional I , u satisfies

$$\begin{aligned} &2\int_\Omega a^{\alpha,\beta}_{i,j}(u(x),x)D_\alpha u^i(x)D_\beta\phi^j(x)\,dx \\ &+\int_\Omega a^{\alpha,\beta}_{i,j,k}(u(x),x)D_\alpha u^i(x)D_\beta u^j(x)\phi^k(x)\,dx = 0 \end{aligned} \tag{2.3}$$

$$for \quad \forall\phi(x) \in \overset{\circ}{H}{}^{1,2}(\Omega; R^N).$$

Next, let δ be a positive number satisfying $\delta < \frac{1}{8}dist(\tilde{\Omega}, \partial\Omega)$. For each number h ($0 < h < \delta$) , the parallel transition along with x_m- axis ($m = 1, \cdots, n$) leads to

$$\begin{aligned} &2\int_\Omega a^{\alpha,\beta}_{i,j}(u(x+he_m),x+he_m)D_\alpha u^i(x+he_m)D_\beta\phi^j(x)\,dx \\ &+\int_\Omega a^{\alpha,\beta}_{i,j,k}(u(x+he_m),x+he_m)D_\alpha u^i(x+he_m)D_\beta u^j(x+he_m)\phi^k(x)\,dx = 0 \end{aligned} \tag{2.4}$$

$$\text{for} \quad \forall\phi(x) \in \overset{\circ}{C}{}^{\infty}(\Omega_\delta; R^N).$$

Then we have

$$\begin{aligned} &2\int_\Omega a^{\alpha,\beta}_{i,j}(u(x),x)D_\alpha\Delta_m u^i(x)D_\beta\phi^j(x)\,dx \\ &= 2\int_\Omega [a^{\alpha,\beta}_{i,j}(x,u(x)) - a^{\alpha,\beta}_{i,j}(u(x+he_m),x+he_m)]D_\alpha u^i(x+he_m)D_\beta\phi^j(x)\,dx \\ &-\int_\Omega a^{\alpha,\beta}_{i,j,k}(x+he_m,u(x+he_m))D_\alpha u^i(x+he_m)D_\beta u^j(x+he_m)\phi^k(x)dx \\ &+\int_\Omega a^{\alpha,\beta}_{i,j,k}(x,u(x))D_\alpha u^i(x)D_\beta u^j(x)\phi^k(x)dx \end{aligned} \tag{2.5}$$

after subtracting (2.4) from (2.3) .

We now substitute $\Delta_m u(x)\zeta^2(x)$ into $\phi(x)$ in (2.5) , where $\zeta(x) \in \overset{\circ}{C}^\infty(\Omega_\delta; R)$ is defined by

$$\zeta(x) = \begin{cases} 1 & : \Omega_{4\delta} \\ 0 & : \Omega / \Omega_{3\delta} \end{cases} \qquad \text{with} \qquad |D\zeta(x)| \le \frac{2}{\delta}. \tag{2.6}$$

Then we have

$$\begin{aligned}
&2 \int_\Omega a_{i,j}^{\alpha,\beta}(u(x),x) D_\alpha(\Delta_m u^i(x)) D_\beta(\Delta_m u^j(x))\, \zeta^2(x)dx \\
&+4 \int_\Omega a_{i,j}^{\alpha,\beta}(u(x),x) D_\alpha(\Delta_m u^i(x)) D_\beta \zeta(x) \Delta_m u^j(x) \zeta(x)\, dx \\
&\quad = 2 \int_\Omega [a_{i,j}^{\alpha,\beta}(x,u(x)) - a_{i,j}^{\alpha,\beta}(u(x+he_m), x+he_m)] D_\alpha u^i(x+he_m) \\
&\qquad [D_\beta(\Delta_m u^j(x))\zeta^2(x) + 2\Delta_m u^j(x) D_\beta\zeta(x)\zeta(x)]dx \\
&- \int_\Omega a_{i,j,k}^{\alpha,\beta}(x+he_m, u(x+he_m)) D_\alpha u^i(x+he_m) D_\beta u^j(x+he_m) \Delta_m u^k(x) \zeta^2(x)dx \\
&+ \int_\Omega a_{i,j,k}^{\alpha,\beta}(x,u(x)) D_\alpha u^i(x) D_\beta u^j(x) \Delta_m u^k(x) \zeta^2(x) dx
\end{aligned} \tag{2.7}$$

Here, we estimate the left-hand side of (2.7) , which we call (L) , from below . First , by using (1.2) , we have

$$\begin{aligned}
(L) &\ge 2\lambda \int_\Omega |D(\Delta_m u(x))|^2 \zeta^2(x) dx \\
&\quad - 4nN\Lambda \int_\Omega |D(\Delta_m u(x))| |\zeta(x)| \Delta_m u(x)| |D\zeta(x)| dx .
\end{aligned} \tag{2.8}$$

Second , applying *the Schwarz inequality* to the second term of (2.8) with $\varepsilon = \frac{\lambda}{2nN\Lambda}$, we have

$$\begin{aligned}
(L) &\ge 2\lambda \int_\Omega |\Delta_m(Du(x))|^2 \zeta^2(x) dx \\
&\quad - 2\varepsilon nN\Lambda \int_\Omega |\Delta_m(Du(x))|^2 \zeta^2(x) dx \\
&\quad - \frac{2nN\Lambda}{\varepsilon} \int_\Omega |D\zeta(x)|^2 |\Delta_m u(x)|^2 dx \\
&\ge \lambda \int_\Omega |\Delta_m(Du(x))|^2 \zeta^2(x) dx \\
&\quad - 2\frac{2(nN\Lambda)^2}{\lambda} \int_\Omega |D\zeta^2(x)| |\Delta_m u(x)|^2 dx .
\end{aligned} \tag{2.9}$$

On the other hand , we perform the estimates of the right-hand side of (2.7) , which we call

(R) .

$$(R) = -2\int_{\Omega}\int_0^1 \frac{da_{i,j}^{\alpha,\beta}}{dt}(x + the_m , u(x) + t\Delta_m u(x))dt\, D_\alpha u^i(x + he_m)$$
$$[D_\beta(\Delta_m u^j(x))\zeta^2(x) + 2\Delta_m u^j(x) D_\beta\zeta(x)\zeta(x)]dx$$
$$- \int_{\Omega} a_{i,j,k}^{\alpha,\beta}(x + he_m, u(x + he_m)) D_\alpha u^i(x + he_m) D_\beta u^j(x + he_m)\Delta_m u^k(x)\zeta^2(x)dx$$
$$+ \int_{\Omega} a_{i,j,k}^{\alpha,\beta}(x, u(x)) D_\alpha u^i(x) D_\beta u^j(x)\Delta_m u^k(x)\zeta^2(x)dx \tag{2.10}$$

By using (1.4) and the boundedness of u , and applying *the Schwarz inequality* to (2.10) , we have

$$(R) \le C_1 \int_{\Omega}(h + |\Delta_m u(x)|)\,\zeta(x)dx\,.$$
$$+ C_1 \int_{\Omega}(h + |\Delta_m u(x)| + |\Delta_m u(x)|^2)[\,|Du(x + he_m)|^2 + |Du(x)|^2\,]\zeta(x)^2\, dx\,. \tag{2.11}$$

Thus , by combining (2.9) with (2.11) , we have the following :

$$\int_{\Omega_{4\delta}} |\Delta_m(Du(x))|^2 dx \le C_2 \int_{\Omega_{3\delta}} |\Delta_m u(x)|^2 dx$$
$$+ C_2 \int_{\Omega_{3\delta}} (h + |\Delta_m u(x)|)dx$$
$$+ C_2 \int_{\Omega_{3\delta}} (h + |\Delta_m u(x)|)[|Du(x)|^2 dx + |Du(x + he_m)|^2]dx \tag{2.12}$$

Here , it is a well-known fact that a minimizer u satisfies *a so-called Caccioppoli inequality* (see [Gm]) : There exists a positive constant C , depending only on n , N , λ, Λ , M such that

$$\int_{B_R} |Du(x)|^2 dx \le \frac{C}{R^2}\int_{B_{2R}} |u(x) - u_R|^2 dx \tag{2.13}$$

holds for any ball $B_{2R} \subset\subset \Omega$ with $0 < {}^\forall R < \delta$. A direct application of the above inequality to *Gering inequality* due to *F.W.Gering* [Ge] (see also [Gm]) leads to the following: There exists a positive number $p\,(p > 2)$, which can be supposed to satisfy $p < 4$ and C depending only on n , N , λ , Λ, Ω , M such that $Du(x)$ belongs to $L^p_{loc}(\Omega\,; R^N)$ and moreover

$$\left(\frac{1}{|\tilde\Omega|}\int_{\tilde\Omega} |Du(x)|^p dx\right)^{\frac{1}{p}} \le \left(\frac{1}{|\Omega|}\int_{\Omega} |Du(x)|^2 dx\right)^{\frac{1}{2}} \tag{2.14}$$

holds for ${}^\forall\tilde\Omega \subset\subset \Omega$.

Thus , we apply *Hölder inequality* to the second term of the right-hand side of (2.12) and we have

$$\int_{\Omega_{4\delta}} |\Delta_m(Du(x))|^2 dx \le C_4 \int_{\Omega_{2\delta}} [h + |\Delta_m u(x)| + |\Delta_m u(x)|^2]dx$$
$$+ C_4 \left[\int_{\Omega_{2\delta}} |D_m u(x)|^p dx\right]^{\frac{2}{p}} \left[\int_{\Omega_{2\delta}} |\Delta_m u(x)|^{\frac{p}{p-2}} dx\right]^{\frac{p-2}{p}}\,. \tag{2.15}$$

In addition , by using (2.13),(2.14) and the boundedness of u , we have

$$\int_{\Omega_{4\delta}} |\Delta_m(Du(x))|^2 dx \leq C_4 \int_{\Omega_{2\delta}} [h + |\Delta_m u(x)| + |\Delta_m u(x)|^2] dx$$

$$+ C_5 \left[\int_{\Omega_{2\delta}} |\Delta_m u(x)|^{\frac{p}{p-2}} dx\right]^{\frac{p-2}{p}} . \tag{2.16}$$

Since $2 < p < 4$ implies $\frac{p}{p-2} > 2$ it follows from the boundedness of $u(x)$ that

$$\int_{\Omega_{4\delta}} |\Delta_m(Du(x))|^2 dx \leq C_4 \int_{\Omega_{2\delta}} [h + |\Delta_m u(x)| + |\Delta_m u(x)|^2] dx$$

$$+ C_6 \left[\int_{\Omega_{2\delta}} |\Delta_m u(x)|^2 dx\right]^{\frac{p-2}{p}} . \tag{2.17}$$

Also , from *Newton-Leibnitz formula* and *a Caccioppoli inequality* , we obtain

$$\int_{\Omega_{2\delta}} |\Delta_m u(x)|^2 dx \leq C_7 h^2 \int_{\Omega_\delta} |Du(x)|^2 dx \leq C_8 h^2 . \tag{2.18}$$

Consequently , from (2.17) and (2.18) , we deduce

$$\int_{\Omega_{4\delta}} |\Delta_m(Du(x))|^2 dx \leq C_8 h^{\frac{2}{p}(p-2)} . \tag{2.19}$$

[1] Also , for any fixed unit vector e one can easily prove

$$\int_{\Omega_{4\delta}} |\Delta_e(Du(x))|^2 dx \leq C_8 h^{\frac{2}{p}(p-2)} \tag{2.20}$$

Proof of Theorem 2

From (2.19) , we obtain an equi-continuity of a sequence of minimizers in $H^{1,2}_{loc}(\Omega\,; R^N)$. Also , it follows from (2.14) that a sequence of minimizers satisfies a uniform boundedness in $H^{1,2}_{loc}(\Omega\,; R^N)$. Thus we obtain the assertion of this theorem from *Rellich - Kondrachev theorem* , (see [Ad]) .

Proof of Theorem 3

The proof of this Theorem is based on estimate (2.20) and the following lemma due to [Gi] (see also [Gm]) .

[1] The estimate (2.19) and (2.20) play an important role in the proofs of the Theorem 2 and Theorem 3

LEMMA 3.1.
Let v be a function in $L^1_{loc}(\Omega)$ and β be any number satisfying $n-2\alpha < \beta < n$. Set

$$E_\beta = \{x \in \Omega : \limsup_{\rho \to +0} \rho^{-\beta} \int_{B_\rho(x)} |v(y)|\, dy > 0\}. \tag{2.21}$$

Then , we have

$$H^{(\beta)}(E_\beta) = 0. \tag{2.22}$$

First, to apply Lemma 3.1 to the proof of Theorem 3 we construct a support function defined as follows : For $\rho_k = \delta(\frac{1}{2})^{k+1}$ $(k = 1, 2, \cdots)$ with $\delta = dist(\tilde{\Omega}, \partial\Omega)$ and a sequence $\{e_k\}_{k\geq 1}$ of unit vectors in R^n we define

$$\varphi_k(y) = \rho_k^{-(n-\beta)-\varepsilon}|Du(y+\rho_k e_k) - Du(y)|^2 \quad \text{with} \quad \varepsilon = \frac{1}{2}(2\alpha - (n-\beta)). \tag{2.23}$$

When we set

$$\phi_k(y) = \sum_{j=1}^{k} \varphi_j(y), \tag{2.24}$$

one easily finds that the function $\phi_k(y)$ is a non-decreasing function for k and the following

$$\int_{\tilde{\Omega}} \phi_k(y)dy = \sum_{j=1}^{k} \int_{\tilde{\Omega}} \varphi_j(y)dy$$

$$= \sum_{j=1}^{k} \rho_j^{-(n-\beta)-\varepsilon} \int_{\tilde{\Omega}} |Du(y+\rho_k e_k) - Du(y)|^2 dy \leq C_8 \sum_{j=1}^{k} \rho_j^{2\alpha-(n-\beta)-\varepsilon} \tag{2.25}$$

$$\leq C_8 \sum_{j=1}^{k} \delta^{\frac{1}{2}(2\alpha-(n-\beta))}(\frac{1}{2})^{\frac{j}{2}\{2\alpha-(n-\beta)\}} = C_8 \delta^{\frac{1}{2}(2\alpha-(n-\beta))} \sum_{j=1}^{k} 2^{-\frac{j}{2}\{2\alpha-(n-\beta)\}} \leq C_9 < \infty.$$

follows from (2.20) and assumption of $2\alpha - (n-\beta) = \beta - (n-2\alpha) > 0$.

Thus $\{\phi_k\}_{k\geq 1}$ is a sequence of measurable functions and moreover , putting $\phi_\infty(y) = \lim_{k\to\infty} \phi_k(y)$, we obtain from *Beppo-Levi Theorem*

$$\int_{\tilde{\Omega}} \phi_\infty(y)dy = \lim_{k\to\infty} \int_{\tilde{\Omega}} \phi_k(y)dy \leq C_{10}. \tag{2.26}$$

Consequently , $\phi_\infty(y)$ is an integrable function on $\tilde{\Omega}$ and

$$\varphi_k(y) \leq \phi_\infty(y) \quad \text{for any } k \quad \text{and almost all} \quad y \in \tilde{\Omega}. \tag{2.27}$$

To complete the proof of theorem , it is sufficient to show

$$S \subset E_\beta \quad \text{, namely,} \quad \text{if} \quad x_0 \notin E_\beta, \quad \text{then} \quad x_0 \notin S. \tag{2.28}$$

Now we fix $x_0 \notin E_\beta$, Then we show that the function

$$(2.29) \qquad r \longmapsto (Du)_{x_0,r} = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} (Du)(y)dy$$

is a continuous and bounded function in the open interval $(0,\delta)$ with $\delta = dist(x_0, \partial\Omega)$.

At first , we shall estimate $|(Du)_{x_0,R_i} - (Du)_{x_0,R_{i+1}}|$ $(i = 1, 2, \cdots)$. Also, by integrating the following (2.30) over $B_{R_i}(x_0)$ $R_i = \frac{\delta}{2}(\frac{1}{2})^2$ $(i = 1, , \cdots)$,

$$|(Du)_{x_0,R_i} - (Du)_{x_0,R_{i+1}}|$$

$$(2.30) \qquad \leq |(Du)_{x_0,R_i} - (Du)(x)| + |(Du)_{x_0,R_{i+1}} - (Du)(x)| .$$

we obtain

$$|B_{R_i}| \cdot |Du_{x_0,R_i} - Du_{x_0,R_{i+1}}|$$

$$(2.31) \qquad \leq \int_{B_{R_i}} |Du_{x_0,R_i} - Du(x)|dx + \int_{B_{R_i}} |Du_{x_0,R_{i+1}} - Du(x)|dx .$$

Next, dividing (2.31) by $|B_{R_i}|$ and by using Hölder inequality , we have

(2.32)

$$\begin{aligned}
&|Du_{x_0,R_i} - Du_{x_0,R_{i+1}}| \\
&\leq \frac{1}{|B_{R_i}|}\int_{B_{R_i}} |Du_{x_0,R_i} - Du(x)|dx + \frac{1}{|B_{R_i}|}\int_{B_{R_i}} |Du_{x_0,R_{i+1}} - Du(x)|dx \\
&\leq \frac{1}{|B_{R_i}|}\int_{B_{R_i}} |\frac{1}{|B_{R_i}|}\int_{B_{R_i}} Du(y)\,dy - Du(x)|dx + \frac{1}{|B_{R_i}|}\int_{B_{R_i}} |\frac{1}{|B_{R_{i+1}}|}\int_{B_{R_{i+1}}} Du(y)\,dy - Du(x)|dx \\
&\leq \frac{1}{|B_{R_i}|^2}\int_{B_{R_i}} dx \int_{B_{R_i}} |Du(y) - Du(x)|dy + \frac{1}{|B_{R_i}||B_{R_{i+1}}|}\int_{B_{R_i}} dx \int_{B_{R_{i+1}}} |Du(y) - Du(x)|dy \\
&\leq \frac{1+2^n}{|B_{R_i}|^2}\int_{B_{R_i}} dx \int_{B_{R_i}} |Du(y) - Du(x)|dy. \\
&\leq \frac{1+2^n}{|B_{R_i}|^2}[\int_{B_{R_i}} dx \int_{B_{R_i}} dy]^{\frac{1}{2}}[\int_{B_{R_i}} dx \int_{B_{R_i}} |Du(y) - Du(x)|^2 dy]^{\frac{1}{2}} \\
&\leq \frac{1+2^n}{|B_{R_i}|}[\int_{B_{R_i}} dx \int_{B_{R_i}} |Du(y) - Du(x)|^2 dy]^{\frac{1}{2}} .
\end{aligned}$$

Here, we extend $Du(x)$ to be zero outside B_{R_i} and successively rewrite it to be $Du(x)$ for convenience. Then we continue the estimates of (2.32) as follows : From the change of variables,

$$\begin{cases} \bar{x} = x, \\ \bar{y} = y - x \end{cases}$$

we obtain

$$\frac{(1+2^n)}{|B_{R_i}|}[\int_{B_{R_i}} dx \int_{B_{R_i}} |(Du)(y) - (Du)(x)|^2 dy]^{1/2} tag2.33$$

$$= \frac{(1+2^n)}{|B_{R_i}|}[\int_{B_{R_i}} dx \int_{B_{2R_i}(x)} |(Du)(y) - (Du)(x)|^2 dy]^{1/2}$$

$$= \frac{(1+2^n)}{|B_{R_i}|}[\int_{B_{R_i}} d\bar{x} \int_{B_{2R_i}(0)} |(Du)(\bar{x}+\bar{y}) - (Du)(\bar{x})|^2 dy]^{1/2}.$$

By using *Fubini Theorem* and successively the mean value theorem , there exists a vector $\bar{y}_i^* \in R^n$ with $0 < |\bar{y}_i^*| < 2R_i$ such that

$$(2.33) = [\frac{c_{11}}{|B_{R_i}|}\int_{B_{R_i}} |(Du)(\bar{x}+\bar{y}_i^*) - (Du)(\bar{x})|^2 d\bar{x}]. \tag{2.34}$$

From (2.32) and (2.34) , we obtain

$$|(Du)_{x_0,R_i} - (Du)_{x_0,R_{i+1}}| \le C_{12}\left[\frac{R_i^{n-\beta+\varepsilon}}{|B_{R_i}|}\int_{B_{R_i}} \frac{|(Du)(\bar{x}+\bar{y}_i^*) - (Du)(\bar{x})|^2}{|\bar{y}_i^*|^{n-\beta+\varepsilon}} d\bar{x}\right]^{\frac{1}{2}}. \tag{2.35}$$

Next we shall show that $\{Du_{x_0,r}\}$ $(r > 0)$ is a *Cauchy filter* . Let r and R $(r < R)$ be positive numbers sufficiently small and then we can take positive integer j and i $(i \le j)$ such that $R_{j+1} < r \le R_j$ and $R_{i+1} < R \le R_i$. We estimate $| Du_{x_0,r} - Du_{x_0,R}|$ by dividing it into the following three terms :

$$\begin{aligned} &| Du_{x_0,r} - Du_{x_0,R}| \\ &\le | Du_{x_0,r} - Du_{x_0,R_j}| + | Du_{x_0,R_j} - Du_{x_0,R_i}| + | Du_{x_0,R} - Du_{x_0,R_i}| \end{aligned} \tag{2.36}$$

Thus , by the same way as above , for $0 < r < R < \delta$, the following holds :

$$|(Du)_{x_0,r} - (Du)_{x_0,R}| \le C_{12}\sum_{k=i}^{j} R_k^{\varepsilon/2}\left[R_k^{-\beta}\int_{B_{R_k}} \frac{|(Du)(\bar{x}+\bar{y}_i^*) - (Du)(\bar{x})|^2}{|\bar{y}_i^*|^{n-\beta+\varepsilon}} d\bar{x}\right]^{\frac{1}{2}}. \tag{2.37}$$

We obtain from $\beta - (n-2\alpha) \ge 0$,

$$\sum_{k=i}^{j} R_k^{\frac{\varepsilon}{2}} \le R_i^{\frac{\varepsilon}{2}} \frac{1 - (\frac{\delta}{2})^{(j-i)\frac{\varepsilon}{2}}}{1 - (\frac{\delta}{2})^{\frac{\varepsilon}{2}}}$$

By noting

$$\frac{|(Du)(\bar{x}+\bar{y}_k^*) - (Du)(\bar{x})|^2}{|\bar{y}_k^*|^{n-\beta+\varepsilon}} \le \phi_\infty(x) \qquad a.e \quad \bar{x} \in \tilde{\Omega} \text{ and } k = 1,2,\cdots.$$

we can continue to estimate (2.37) as follows :

$$|(Du)_{x_0,r} - (Du)_{x_0,R}|$$

$$\leq C_{14}R^{\frac{\epsilon}{2}}\left[\underset{k>0}{ess.sup}R_k^{-\beta}\int_{B_{R_k}}\phi_\infty(y)dy\right]^{\frac{1}{2}}. \tag{2.38}$$

Also , from (2.28) , there exists a constant K such that

$$|(Du)_{x_0,r}-(Du)_{x_0,R}|\leq C_{14}K^{\frac{1}{2}}R^{\frac{\beta-(n-2\alpha)}{2}} \tag{2.39}$$

This shows that $\{(Du)_{x_0,r}\}_{r>0}$ is *a Cauchy filter* . Thus, $\lim\limits_{R\to+0}(Du)_{x_0,R}$ surely exists. Also , from (2.39), we obtain

$$|(Du)_{x_0,r}-(Du)_{x_0,R}|\leq C_{14}K^{\frac{1}{2}}R^{\frac{\beta-(n-2\alpha)}{2}}. \tag{2.40}$$

Then

$$\lim_{R\to+0}|(Du)_{x_0,R}|\leq|(Du)_{x_0,\delta/4}|+C_{14}K^{\frac{1}{2}}(\delta/4)^{\frac{\beta-(n-2\alpha)}{2}}. \tag{2.41}$$

Consequently , $\lim\limits_{R\to+0}(Du)_{x_0,R}$ exists and is finite. This shows $x_0\notin S$.

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