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THE DEFINITE INTEGRAL OF PRODUCTS OF BESSEL FUNCTIONS

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I. Introduction

In a problem of the Ising spin glass on the Bethe lattice we encountered a nonlinear integral equation ([1], [2])

$$S(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(x,y)[S(y)]^{z}\, dy \qquad (1.1)$$

$$K(x,y) = 2\pi\delta(y)\cos x - 2j_0(y)\cos x + 2\sum_{m=0}^{\infty}(1+4m)j_{2m}(x)j_{2m}(y) \qquad (1.2)$$

where z is the natural number, and $j_{2m}(x)$ is the spherical Bessel function of order 2m.

We put

$$S(x) = a + b\cos x + \sum_{\ell=0}^{\infty} c_{2\ell}\, j_{2\ell}(x) \quad . \qquad (1.3)$$

Substituting (1.2) and (1.3) into (1.1), we get a system of algebraic equations for unknowns a, b, and $c_{2\ell}$ of which the coefficients are given by definite integral of products of Bessel functions. Solution S(x) can be solved by the solution of the simultaneous algebraic equation. We defined the following integral I:

$$I^{(k)}_{\ell_1 \ell_2 \ldots \ell_\nu} \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \cos^k x \, j_{\ell_1}(x) \, j_{\ell_2}(x) \ldots j_{\ell_\nu}(x) \, dx \tag{1.4}$$

Katsura [3], and Katsura and Nishihara [4] calculated $I^{(0)}_{\ell_1 \ell_2 \ell_3}$ and $I^{(1)}_{\ell_1 \ell_2}$ by using the residues and J. E. Kilpatrick, Katsura and Inoue [5] calculated

$$W^{\lambda}_{\nu_1 \nu_2 \ldots \nu_\ell}(a_1, a_2, \ldots, a_\ell)$$

$$\equiv \int_0^{\infty} J_{\nu_1}(a_1 t) \, J_{\nu_2}(a_2 t) \ldots J_{\nu_\ell}(a_\ell t) \, t^{-\lambda} \, dt \tag{1.5}$$

In this paper we calculate the values of following integrals I, N, V, Λ, and R which appeared in the spin glass calculation as slated above by extending their method.

$$I^{(h;k)}_{n_1 n_2 \ldots n_\nu} \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \sin^h(x+k) \, j_{n_1}(x) j_{n_2}(x) \ldots j_{n_\nu}(x) dx \tag{1.5}$$

$$N^{(h;k)}_{m_1 m_2 \ldots m_\mu} \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \sin^h(x+k) \, n_{m_1}(x) n_{m_2}(x) \ldots n_{m_\mu}(x) dx \tag{1.6}$$

$$V^{(h_1, h_2, \ldots, h_\lambda; k_1, k_2, \ldots, k_\lambda)}_{n_1 n_2 \ldots n_\nu; m_1 m_2 \ldots m_\mu}$$

$$\equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \sin^{h_1}(x+k_1) \sin^{h_2}(x+k_2) \ldots \sin^{h_\lambda}(x+k_\lambda)$$

$$\times \, j_{n_1}(x) j_{n_2}(x) \ldots j_{n_\nu}(x) n_{m_1}(x) n_{m_2}(x) \ldots n_{m_\mu}(x) dx \tag{1.7}$$

$$\Lambda(\ell, m, n) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \{j_0(x)\}^{\ell} \sin^m x \cos^n x \, dx \tag{1.8}$$

$$R^{(m)}_{n_1 n_2 \ldots n_\nu} \equiv \frac{1}{\pi}\int_{-\infty}^{\infty} e^{imx}\, j_{n_1}(x)\, j_{n_2}(x) \ldots j_{n_\nu}(x)\, dx \tag{1.9}$$

where $n_m(x)$ is the spherical Neumann function of order m.

II. The integral for I, N, V, Λ, and R

$j_n(x)$ and $n_m(x)$ can be expressed in

$$j_n(x) = \frac{1}{2x}\sum_{r=0}^{n} [A_{nr}e^{ix} + A^*_{nr}e^{-ix}]\, x^{-r} \tag{2.1}$$

$$n_m(x) = \frac{1}{2x}\sum_{s=0}^{m} (-1)^{m+1} [B_{ms}e^{ix} + B^*_{ms}e^{-ix}]\, x^{-s} \tag{2.2}$$

$$A_{nr} = Y_{nr}\, i^{r-n-1}, \quad B_{ms} = Y_{ms}\, i^{m+s},$$

$$Y_{nr} = \frac{(n+r)!}{n!(n-r)!}\, 2^{-r} \tag{2.3}$$

(A^*_{nr} is the comprex conjugate of A_{nr}) and

$$\sin^h(x+k) = \sum_{p=0}^{h}\sum_{q=0}^{p}\sum_{u=0}^{h-p} \cos^p k\, \sin^{h-p} k\, (-1)^{(p+2q)/2}\, 2^{-h}$$
$$\times \binom{h}{p}\binom{p}{q}\binom{h-p}{u} \exp(i(h-2q-2u)x). \tag{2.4}$$

Furthermore by using

$$P\int_{-\infty}^{\infty} \frac{\exp(iky)}{y^\ell}\, dy = \frac{(ik)^{\ell-1}}{(\ell-1)!}\, i\pi\, \mathrm{sgn}(k) \quad (k\neq 0) \tag{2.5}$$

where P is the principal part. Hence by using (2.1)-(2.5), we can evaluate them.

$$I^{(h;k)}_{n_1 n_2 \ldots n_\nu} = \sum_{p=0}^{h} \sum_{q=0}^{p} \sum_{u=0}^{h-p} \sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} \cdots \sum_{r_\nu=0}^{n_\nu} \sum_{\phi=0}^{\nu} \cos^h k$$

$$\times \sin^{h-p} k \binom{h}{p} \binom{p}{q} \binom{h-p}{u} \frac{1}{2^{h+\nu}}$$

$$\times \frac{\prod_{\omega=1}^{\nu} Y_{n_\omega r_\omega}}{\left(\sum_{\omega=1}^{\nu} r_\omega + \nu - 1\right)!} (-1)^{\frac{1}{2}\left\{\sum_{\omega=1}^{\nu}(2r_\omega + n_\omega) + (p+2u)\right\}}$$

$$\times W_\phi(\nu, 0)\ f^{\sum_{\omega=1}^{\nu} r_\omega + \nu - 1}\ \mathrm{sgn}(f) \qquad (2.6)$$

$$N^{(h;k)}_{m_1 m_2 \ldots m_\mu} = \sum_{p=0}^{h} \sum_{q=0}^{p} \sum_{u=0}^{h-p} \sum_{s_1=0}^{m_1} \sum_{s_2=0}^{m_2} \cdots \sum_{s_\nu=0}^{m_\mu} \sum_{\phi=0}^{\mu} \cos^h k$$

$$\times \sin^{h-p} k \binom{h}{p} \binom{p}{q} \binom{h-p}{u} \frac{1}{2^{h+\mu}}$$

$$\times \frac{\prod_{\omega=1}^{\mu} Y_{m_\omega s_\omega}}{\left(\sum_{\omega=1}^{\mu} s_\omega + \mu - 1\right)!} (-1)^{\frac{1}{2}\left\{\sum_{\omega=1}^{\mu}(2s_\omega + m_\omega + \mu) + (p+2u)\right\}}$$

$$\times W_\phi(0, \mu)\ g^{\sum_{\omega=1}^{\mu} s_\omega + \mu - 1}\ \mathrm{sgn}(g) \qquad (2.7)$$

(where if μ is odd, the integral for N = 0, because N is the odd function)

$$V^{(h_1, h_2, \ldots, h_\lambda; k_1, k_2, \ldots k_\lambda)}_{n_1 n_2 \ldots n_\nu; m_1 m_2 \ldots m_\mu}$$

$$= \sum_{p_1=0}^{h_1} \sum_{p_2=0}^{h_2} \cdots \sum_{p_\lambda=0}^{h_\lambda} \sum_{q_1=0}^{p_1} \sum_{q_2=0}^{p_2} \cdots \sum_{q_\lambda=0}^{p_\lambda} \sum_{u_1=0}^{h_1-p_1} \sum_{u_2=0}^{h_2-p_2} \cdots$$

$$\sum_{u_\lambda=0}^{h_\lambda-p_\lambda} \sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} \cdots \sum_{r_\nu=0}^{n_\nu} \sum_{s_1=0}^{m_1} \sum_{s_2=0}^{m_2} \cdots \sum_{s_\mu=0}^{m_\mu} \sum_{\phi=0}^{\nu+\mu}$$

$$\times \prod_{\zeta=1}^{\lambda} \cos^{p_\zeta} k_\zeta \sin^{h_\zeta-p_\zeta} k_\zeta \binom{h_\zeta}{p_\zeta}\binom{p_\zeta}{q_\zeta}\binom{h_\zeta-q_\zeta}{u_\zeta}$$

$$\times \frac{1}{2^{\sum_{\xi=1}^{\lambda} h_\xi+\nu+\mu}} \frac{\prod_{\omega=1}^{\lambda} \prod_{\eta=1}^{\mu} Y_{n_\omega r_\omega} Y_{m_\eta s_\eta}}{(\sum_{\omega=1}^{\nu} r_\omega + \sum_{\eta=1}^{\mu} s_\eta + \nu+\mu-1)!}$$

$$\times (-1)^{\frac{1}{2}\{\sum_{\xi=1}^{\lambda}(p_\xi+2u_\xi)+\sum_{\omega=1}^{\nu}(n_\omega+2r_\omega)+\sum_{\eta=1}^{\mu}(m_\eta+s_\eta+\mu)\}}$$

$$\times W_\phi(\nu,\mu)\ \Phi^{(\sum_{\omega=1}^{\nu} r_\omega + \sum_{\eta=1}^{\mu} s_\eta + \nu+\mu-1)} \operatorname{sgn}(\Phi) \qquad (2.8)$$

$$\Lambda(\ell,m,n) = \sum_{u=0}^{\ell} \sum_{v=0}^{m} \sum_{w=0}^{n} \binom{\ell}{u}\binom{m}{v}\binom{n}{w} \frac{(-1)^{-m/2+u+v}}{2^{\ell+m+n}\ (\ell-1)!}$$

$$\times \Gamma^{\ell-1} \operatorname{sgn}(\Gamma) \qquad (2.9)$$

(where if m is odd, $\Lambda(\ell,m,n) = 0$, because $\Lambda(\ell,m,n)$ is the odd function)

$$R^{(m)}_{n_1 n_2 \ldots n_\nu} = \sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} \cdots \sum_{r_\nu=0}^{n_\nu} \sum_{\phi=0}^{\nu} 2^{-\nu}$$

$$\times (-1)^{\sum_{\omega=1}^{\nu} (2r_\omega+n_\omega)/2} \frac{\prod_{\omega=1}^{\nu} Y_{n_\omega r_\omega}}{(\sum_{\omega=1}^{\nu} r_\omega+\nu-1)!} W_\phi(\nu,0)$$

$$\times \Upsilon^{\sum_{\omega=1}^{\nu} r_\omega + \nu - 1} \operatorname{sgn}(\Upsilon) \qquad (2.10)$$

where $f\equiv(h+\nu)-2(q+u+\phi)$, $g\equiv(h+\mu)-2(q+u+\phi)$, $\Phi\equiv(\sum_{\eta=1}^{\lambda} h_\eta+\nu+\mu) -2\{\sum_{\eta=1}^{\lambda}(q_\eta+u_\eta)+\phi\}$, $\Gamma\equiv(\ell+m+n)-2(u+v+w)$, and $\Upsilon\equiv m+\nu-2\phi$. $W_\phi(\nu,\mu)$ is as follows.

$$W_0(\nu,\mu) \equiv 1 \tag{2.11}$$

$$W_1(\nu,\mu) \equiv \sum_{\xi=1}^{\nu} (-1)^{r_\xi+n_\xi+1} + \sum_{\eta=1}^{\mu} (-1)^{s_\eta+m_\eta} \tag{2.12}$$

$W_\phi(\nu,\mu)$ ($\phi = 2,\ldots,\nu$) is defined by the sum of $\binom{\nu+\mu}{\phi}$ products of ϕ pieces of $(-1)^{r_\lambda+n_\lambda+1}$ $(\lambda=1,\ldots,\nu)$ and $(-1)^{s_\zeta+m_\zeta}$ $(\zeta=1,\ldots,\mu)$.

Example. $\nu = 3$, $\mu = 2$, $\phi = 4$

$$\begin{aligned} W_4(3,2) &= (-1)^{(r_1+r_2+r_3+s_1)+(n_1+n_2+n_3+m_1)+3} \\ &+ (-1)^{(r_1+r_2+r_3+s_2)+(n_1+n_2+n_3+m_2)+3} \\ &+ (-1)^{(r_1+r_2+s_1+s_2)+(n_1+n_2+m_1+m_2)+2} \\ &+ (-1)^{(r_1+r_3+s_1+s_2)+(n_1+n_3+m_1+m_2)+2} \\ &+ (-1)^{(r_2+r_3+s_1+s_2)+(n_2+n_3+m_1+m_2)+2} \end{aligned} \tag{2.13}$$

$W_\phi(\nu,0)$ ($W_\phi(0,\mu)$) is defined only of $(-1)^{r+n+1}$ ($(-1)^{m+s}$).

Values of I and N for $2 \leqq \nu$ $(\mu) \leqq 6$, n_ν $(m_\mu) = 0, 2$, and $0 \leqq h \leqq 4$ are shown as tables in Appendix. Values of V for $\lambda = 1$, $\nu = 1, 2$, $\mu=1$, $0 \leqq n_\nu \leqq 3$, $m_\mu = 0$, and $0 \leqq h_\lambda \leqq 1$ are also in Appendix.

Reference

[1] S. Katsura, Physica 141A (1987) 556; 149A (1988) 371.

[2] S. Katsura, Prog. Theor. Phys. Suppl. 87 (1986) 139;

[3] J. E. Kilpatrick, S. Katsura, and Y. Inoue, Mathematics of Computation 21 NO.99 (1967) 407.

[4] S. Katsura, Phys. Rev. 115 (1959) 1417.

[5] S. Katsura and K. Nishihara, J. Chem. Phys. 50 (1967) 3579.

Appendix Tables of the integral of products of Bessel functions

Here tables for I in Tables 1.1-1.3, N in Tables 2.1-2.3, V in Tables 3.1-3.2 are shown.

Table 1.1 The integral for I ($h=0,1$, $\nu=2,\dots,6$, $n_\nu=0,2$, $k=\pi/2$)

$n_1n_2n_3n_4n_5n_6$ h	0	1
0 0	1	$\frac{1}{2}$
0 2	0	0
2 2	$\frac{1}{5}$	$-\frac{7}{80}$
0 0 0	$\frac{3}{4}$	$\frac{1}{2}$
0 0 2	$\frac{1}{16}$	0
0 2 2	$\frac{1}{160}$	0
2 2 2	$\frac{9}{256}$	$-\frac{1}{35}$
0 0 0 0	$\frac{2}{3}$	$\frac{23}{48}$
0 0 0 2	$\frac{1}{30}$	$\frac{1}{120}$
0 0 2 2	$\frac{1}{105}$	$-\frac{31}{13440}$
0 2 2 2	0	$-\frac{3}{17920}$
2 2 2 2	$\frac{4}{385}$	$-\frac{6037}{788480}$
0 0 0 0 0	$\frac{115}{192}$	$\frac{11}{24}$
0 0 0 0 2	$\frac{49}{1920}$	$\frac{1}{120}$
0 0 0 2 2	$\frac{123}{35840}$	0
0 0 2 2 2	$\frac{121}{71680}$	$-\frac{1}{1400}$
0 2 2 2 2	$-\frac{37}{1892352}$	$\frac{1}{23100}$
2 2 2 2 2	$\frac{6569}{2408448}$	$-\frac{2377}{1051050}$
0 0 0 0 0 0	$\frac{11}{20}$	$\frac{841}{1920}$
0 0 0 0 0 2	$\frac{2}{105}$	$\frac{9}{1190}$
0 0 0 0 2 2	$\frac{1}{420}$	$\frac{103}{645120}$
0 0 0 2 2 2	$\frac{1}{2200}$	$-\frac{919}{11827200}$
0 0 2 2 2 2	$\frac{1}{2730}$	$-\frac{275}{1490944}$
0 2 2 2 2 2	$-\frac{151}{3503500}$	$\frac{1362967}{64576512000}$
2 2 2 2 2 2	$\frac{3331}{4254250}$	$-\frac{2165020841}{334567833600}$

Table 1.2 The integral for I (h=2,3, ν=2,...,6, n_ν=0,2, k=π/2)

$n_1n_2n_3n_4n_5n_6$ h	2	3
0 0	$\frac{1}{2}$	$\frac{3}{8}$
0 2	0	0
2 2	$\frac{1}{10}$	$-\frac{21}{320}$
0 0 0	$\frac{7}{16}$	$\frac{3}{8}$
0 0 2	$\frac{1}{64}$	0
0 2 2	$\frac{1}{640}$	0
2 2 2	$\frac{811}{35840}$	$-\frac{3}{140}$
0 0 0 0	$\frac{5}{12}$	$\frac{35}{96}$
0 0 0 2	$\frac{1}{120}$	$\frac{1}{240}$
0 0 2 2	$\frac{1}{420}$	$-\frac{31}{26880}$
0 2 2 2	0	$-\frac{3}{35840}$
2 2 2 2	$\frac{53}{7700}$	$-\frac{6707}{1126400}$
0 0 0 0 0	$\frac{51}{128}$	$\frac{17}{48}$
0 0 0 0 2	$\frac{9}{1280}$	$\frac{1}{240}$
0 0 0 2 2	$\frac{41}{71680}$	0
0 0 2 2 2	$\frac{53}{102400}$	$-\frac{1}{2800}$
0 2 2 2 2	$-\frac{381}{31539200}$	$\frac{1}{46200}$
2 2 2 2 2	$\frac{11210223}{5740134400}$	$-\frac{1871}{1051050}$
0 0 0 0 0 0	$\frac{23}{60}$	$\frac{1761}{5120}$
0 0 0 0 0 2	$\frac{1}{168}$	$\frac{109}{26880}$
0 0 0 0 2 2	$\frac{1}{2520}$	$\frac{103}{1720320}$
0 0 0 2 2 2	$\frac{1}{13200}$	$-\frac{919}{31539200}$
0 0 2 2 2 2	$\frac{1}{15015}$	$-\frac{189269}{1968046080}$
0 2 2 2 2 2	$-\frac{1091}{63063000}$	$\frac{396367}{34440806400}$
2 2 2 2 2 2	$\frac{103013}{178678500}$	$\frac{32493878367}{62452662272000}$

Table 1.3 The integral for I (h=4, ν=2, ..., 6, n_ν=0,2, k=π/2)

$n_1n_2n_3n_4n_5n_6$ h	4
0 0	$\frac{3}{8}$
0 2	0
2 2	$\frac{3}{40}$
0 0 0	$\frac{11}{32}$
0 0 2	$\frac{1}{128}$
0 2 2	$\frac{1}{1280}$
2 2 2	$\frac{1307}{71680}$
0 0 0 0	$\frac{1}{3}$
0 0 0 2	$\frac{1}{240}$
0 0 2 2	$\frac{1}{840}$
0 2 2 2	0
2 2 2 2	$\frac{43}{7700}$
0 0 0 0 0	$\frac{995}{3072}$
0 0 0 0 2	$\frac{113}{30720}$
0 0 0 2 2	$\frac{123}{573440}$
0 0 2 2 2	$\frac{1621}{5734400}$
0 2 2 2 2	$-\frac{5933}{756940800}$
2 2 2 2 2	$\frac{222940799}{137763225600}$
0 0 0 0 0 0	$\frac{101}{320}$
0 0 0 0 0 2	$\frac{11}{3360}$
0 0 0 0 2 2	$\frac{1}{6720}$
0 0 0 2 2 2	$\frac{1}{35200}$
0 0 2 2 2 2	$\frac{37}{480480}$
0 2 2 2 2 2	$-\frac{19}{1848000}$
2 2 2 2 2 2	$\frac{17669}{36652000}$

Table 2.1 The integral for N (h=0,1, μ=2,...,6, m_μ=0,2, $k=\pi/2$)

$m_1m_2m_3m_4m_5m_6$ h	0	1
0 0	-1	$-\frac{3}{2}$
0 2	0	$\frac{3}{2}$
2 2	$-\frac{1}{5}$	$-\frac{39}{80}$
0 0 0 0	2	$\frac{45}{16}$
0 0 0 2	$-\frac{11}{10}$	$-\frac{45}{16}$
0 0 2 2	$\frac{13}{35}$	$\frac{3123}{4480}$
0 2 2 2	0	$-\frac{639}{2560}$
2 2 2 2	$-\frac{12}{385}$	$\frac{811287}{3942400}$
0 0 0 0 0 0	$-\frac{15}{4}$	$-\frac{2191}{384}$
0 0 0 0 0 2	$\frac{18}{7}$	$\frac{2191}{384}$
0 0 0 0 2 2	$-\frac{107}{140}$	$-\frac{275197}{129024}$
0 0 0 2 2 2	$\frac{741}{3080}$	$\frac{1004981}{1182720}$
0 0 2 2 2 2	$-\frac{229}{2002}$	$-\frac{30954931}{1230028800}$
0 2 2 2 2 2	$\frac{24747}{700700}$	$\frac{66527233}{1986969600}$
2 2 2 2 2 2	$\frac{6393}{850850}$	$-\frac{83762843347}{4683949670400}$

Table 2.2 The integral for N ($h=2,3$, $\mu=2,\ldots,6$, $m_\mu=0,2$, $k=\pi/2$)

$m_1m_2m_3m_4m_5m_6$ h	2	3
0 0	$-\frac{3}{2}$	$-\frac{15}{8}$
0 2	3	$\frac{21}{4}$
2 2	$-\frac{3}{10}$	$-\frac{147}{64}$
0 0 0 0	$\frac{15}{4}$	$\frac{455}{96}$
0 0 0 2	$\frac{45}{8}$	$-\frac{1849}{192}$
0 0 2 2	$\frac{303}{140}$	$\frac{4579}{768}$
0 2 2 2	$-\frac{36}{35}$	$-\frac{41403}{17920}$
2 2 2 2	$\frac{369}{1540}$	$\frac{927659}{1576960}$
0 0 0 0 0 0	$-\frac{49}{6}$	$-\frac{11445}{1024}$
0 0 0 0 0 2	$\frac{263}{24}$	$\frac{3653}{192}$
0 0 0 0 2 2	$-\frac{2881}{504}$	$-\frac{669925}{49125}$
0 0 0 2 2 2	$\frac{8293}{3696}$	$\frac{35663743}{6307840}$
0 0 2 2 2 2	$-\frac{20407}{30030}$	$-\frac{855175591}{393609216}$
0 2 2 2 2 2	$\frac{485993}{1801800}$	$\frac{4084660117}{4920115200}$
2 2 2 2 2 2	$-\frac{132823}{1021020}$	$-\frac{2499231739269}{12490532454400}$

Table 2.3 The integral for N ($h=4$, $\mu=2,\ldots,6$, $m_\mu=0,2$, $K=\pi/2$)

$m_1m_2m_3m_4m_5m_6$ h	4
0 0	$-\frac{15}{8}$
0 2	$\frac{15}{2}$
2 2	$-\frac{51}{8}$
0 0 0 0	$\frac{35}{6}$
0 0 0 2	$-\frac{721}{48}$
0 0 2 2	$\frac{329}{24}$
0 2 2 2	$-\frac{39}{7}$
2 2 2 2	$\frac{3659}{1540}$
0 0 0 0 0 0	$-\frac{945}{64}$
0 0 0 0 0 2	$\frac{981}{32}$
0 0 0 0 2 2	$-\frac{1849}{64}$
0 0 0 2 2 2	$\frac{19831}{1408}$
0 0 2 2 2 2	$-\frac{194011}{32032}$
0 2 2 2 2 2	$\frac{680133}{320320}$
2 2 2 2 2 2	$-\frac{134641}{1944800}$

Table 3.1 The integral for V (λ=1, ν,μ=1, n_ν=0, 1, 2, 3, m_μ=0, h_λ=0,1)

$n_1;m_1$ \ h_1	0		1	
	$k = 0$	$k = \frac{\pi}{2}$	$k = 0$	$k = \frac{\pi}{2}$
0 0	0	0	$-\frac{1}{2}$	0
1 0	0	0	0	$-\frac{1}{4}$
2 0	0	0	0	0
3 0	0	0	0	$-\frac{1}{16}$

Table 3.2 The integral for V (λ=1, ν=2, μ=1, n_ν=0, 1, 2, 3, m_μ=0, h_λ=0, k=0 or π/2)

$n_1 n_2;m_1$ \ h_1	0
0 0 0	0
0 1 0	$-\frac{1}{6}$
0 2 0	0
0 3 0	0
1 1 0	0
1 2 0	$\frac{7}{240}$
1 3 0	0
2 2 0	0
2 3 0	$\frac{19}{1120}$
3 3 0	0