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Convergence of an Integral Equation Method to Convective Heat Transfer

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Abstract.

A boundary-domain (or hybrid) integral equation method is applied to the approximate solution of transient convection dominated conduction problems in three dimensions. The domain of interest has non-smooth surface of the Wendland type. Given field velocity is assumed to be non-uniform. Neumann boundary condition is imposed to the problem. Under some conditions which are not much restrictive in practical applications in engineering, the integral equation is proved to be uniquely solvable in the Banach space of continuous functions on the enclosure of the domain with the supremum norm. It is shown as a direct consequence of the Krasnosel'skii's result that the computational scheme by the Galerkin method is inversely stable and approximate solutions converge uniformly to the exact solution.

1. Introduction

A heat transfer problem to be considered in this paper is loosely stated as follows: Given the flow velocity $\mathbf{v}(x) = (v_1, v_2, v_3)$ in a domain Ω in three dimensions with rectangular coordinates such that the incompressibility condition:

$$\frac{\partial v_j}{\partial x_j} = 0 \qquad \text{in } \Omega, \quad t > 0 \tag{1}$$

is satisfied, find unknown temperature $u(x,t)$ which satisfies the transient heat convection conduction equation:

$$\frac{\partial u}{\partial t} + v_j \frac{\partial u}{\partial x_j} = \lambda \Delta u \qquad \text{in } \Omega, \quad t > 0 \tag{2}$$

subject to the boundary and initial conditions:

$$q := -\lambda \frac{\partial u}{\partial n} + v_j n_j u = \hat{q} \qquad \text{on } \Gamma = \partial\Omega, \quad t > 0 \tag{3}$$

and

$$u(x, 0) = u^0(x) \qquad \text{in } \Omega \tag{4}$$

for given total flux $q^\wedge$ on the boundary and given initial temperature distribution $u0(x)$ with given constant heat conductivity $\lambda > 0$ and the exterior normal $n = (n_1, n_2, n_3)$ to the boundary Γ. The proper setting of the problem is presented in the next section.

We shall consider the case that the boundary Γ is a non-smooth surface of some general kind and correspondingly we assume that the given total flux is not bounded on the boundary. The solution will be found in the space of continuous functions on the closure of the domain.

Transient heat conduction problem with Neumann boundary condition on non-smooth surface was considered by Costabel *et al.* [1987] and Onishi[1987]. They showed the unique existence of the solution of corresponding Volterra-Fredholm integral equation of the second kind and presented the uniform convergence of Galerkin approximate solutions. The present paper is the extension of those previous two papers by including the convection effect to the heat conduction problem. Owing to the presence of the convection term with variable field velocities, the integral equation is no longer of the boundary type.

A boundary–domain integral equation approach for the Neumann problem of steady convection–diffusion problem was considered by Onishi[1987], in which the existence of the continuous solution is proved at all Peclet numbers.

2. Integral Equation of the Second Kind

We shall derive an integral equation corresponding to the initial–boundary value problems (1)–(4). To this end, we start with the specification of the domain in question. Let Ω be a simply connected and bounded open domain in $\mathbf{R}^3$. The boundary Γ is assumed to be a piecewise Ljapunow surface. This means that the surface is locally Hoelder continuous with the index $1+\kappa$ ($0 < \kappa \leq 1$). We denote the set of non–smooth points on the surface by γ. It forms edges and corners.

Let $d\Gamma(y)$ be an infinitesimal surface area at the point $y \in \Gamma - \gamma$. The infinitesimal solid angle at x ($\in \mathbf{R}^3$) subtending the area $d\Gamma(y)$ is given by the expression:

$$d\Theta_x(y) := \frac{\partial}{\partial n(y)}\left(\frac{1}{r}\right) d\Gamma(y) \tag{5}$$

with the distance $r = |y-x|$, see Figure 1.

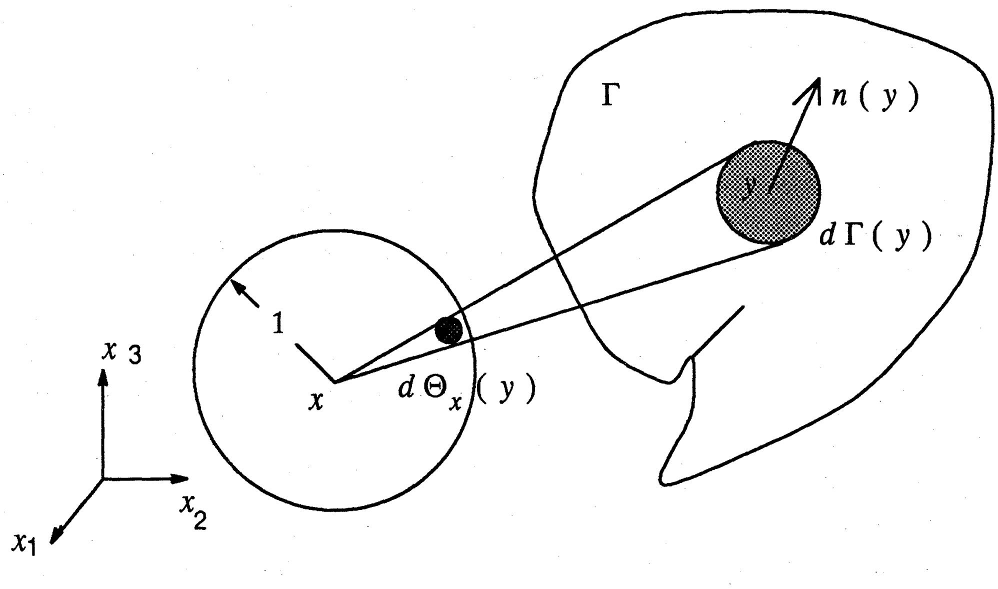

Figure 1. Infinitesimal solid angle.

Under the assumption that the boundary is a piecewise Ljapunow surface, we see that

$$\sup_{x \in \mathbf{R}^3} \int_\Gamma | \, d\Theta_x(y) \, | =: A < +\infty \tag{6}$$

with some constant *A*. The solid angle at x subtending to the whole geometry Γ is given by the expression:

$$\Theta(x) := \int_{y \in \Gamma} d\Theta_x(y) = \begin{cases} 4\pi & (x \in \Omega^\circ) \\ 0 & (x \in \Omega^{ext}) \end{cases} \tag{7}$$

We assume moreover that the boundary Γ satisfies the inequality:

$$\lim_{\delta \to 0} \ \sup_{x \in \Gamma} \ \hat{W}_\delta(x) =: \omega < 1 \tag{8}$$

with some constant ω, in which

$$\hat{W}_\delta(x) := \frac{1}{4\pi} \left\{ \int_{0 < | y - x | \leq \delta} | \, d\Theta_x(y) \, | + | \, 4\pi - \Theta(x) \, | \right\} \tag{9}$$

The piecewise Ljapunow surface satisfying the property (8) is called *quasi–Wendland surface*. We notice that the constant 4π in (9) is replaced by 2π for the integral equation defined only on the boundary as discussed in Wendland[1968]. The assumption (8) implies the inequality $4\pi(1-\omega) \leq \Theta(x)$.

Let the Neumann data $\hat{q}(\,.,t\,)$ be in the space of *p*th–power summable functions $Lp(\Gamma)$ with $p > 2$. We assume that

$$\| \hat{q}(\,.\,,t\,) \|_p := \left\{ \int_\Gamma | \, \hat{q}(x,t) \, |^p \, d\Gamma \right\}^{1/p} \leq M_1 \tag{10}$$

uniformly for all $t \in [0,T]$ with some constant M_1. The boundary condition (3) is understood in the sense of the boundary flow; see Onishi[1986].

As in Costabel *et al.*[1986], we can see that the solution $u(x,t)$ of the initial–boundary value problem at interior point x ($\in \Omega^\circ$) is expressed in the form:

$$u(x,t) = -\lambda \int_0^t d\tau \int_\Gamma u(y,\tau)\, h^*(y,\tau:x,t)\, d\Theta_x(y)$$

$$+ \int_0^t d\tau \int_\Omega u(y,\tau)\, v_j(y,\tau) \frac{\partial u^*}{\partial x_j}(y,\tau:x,t)\, d\Omega \tag{11}$$

$$- \int_0^t d\tau \int_\Gamma \hat{q}(y,\tau)\, u^*(y,\tau:x,t)\, d\Gamma + \int_0^t d\tau \int_\Omega \overset{\circ}{u}(y)\, u^*(y,0:x,t)\, d\Omega(y),$$

where u^* is the fundamental solution to the heat operator, *i.e.*,

$$\frac{\partial u^*}{\partial \tau} + \lambda \Delta_y u^* = -\delta(y-x)\,\delta(t-\tau) \quad , \tag{12}$$

$$u^* = \begin{cases} \left(\dfrac{1}{2\sqrt{\pi\lambda(t-\tau)}}\right)^3 \exp\left[-\dfrac{r^2}{4\lambda(t-\tau)}\right] & (t>\tau) \\ 0 & (t<\tau) \end{cases} \tag{13}$$

and

$$h^*(y,\tau:x,t) = \frac{r^3}{2\lambda(t-\tau)}\, u^*(y,\tau:x,t) \quad . \tag{14}$$

We notice that all integrals involved in (11) are weakly singular in the sense that they are absolutely convergent. This nice property is due to the assumption that the surface is piecewise Ljapunow. We shall show here the weak singularity only for the first integral on the right hand side of (11). In fact, on each subsurface Γ_i, the integral is written in the form:

$$\int_0^t d\tau \int_{\Gamma_i} u(y,\tau)\, h^*(y,\tau:x,t)\, d\Theta_x(y)$$

$$= \int_0^t d\tau \int_{\Gamma_i} u(y,\tau) \frac{\partial u^*}{\partial n}(y,\tau:x,t)\, d\Gamma$$

where the normal derivative is calculated as follows.

$$\frac{\partial u^*}{\partial n}(y,\tau:x,t)$$

$$=\left(\frac{1}{2\sqrt{\pi\lambda(t-\tau)}}\right)^3\left[\frac{-r}{2\lambda(t-\tau)}\right]\exp\left[\frac{-r^2}{4\lambda(t-\tau)}\right]\frac{y_j-x_j}{r}n_j(y)\ .$$

Since Γ_i is a Ljapunow surface, it follows that

$$\left|\frac{y_j-x_j}{r}n_j(y)\right|=|\cos\nu|\ \leq\ L(\Gamma)\,|y-x|^{\kappa}$$

for the angle ν between two vectors $y-x$ and $n(y)$ with the constant L. Using the inequality $\xi^s e^{-\xi}\leq s^s e^{-s}\ (s>0)$, we can see that

$$\left|\frac{\partial u^*}{\partial n}(y,\tau:x,t)\right|\ \leq\frac{G_1}{(t-\tau)^{\mu}}\ \frac{L(\Gamma)}{r^{4-2\mu-\kappa}}$$

for all $\mu<1$ with some constant G_1. Choose μ so that $4-2\mu-\kappa<2$. This implies that $1-\kappa/2<\mu<1$ and the integral is absolutely convergent.

As regard to the continuity of the second and third integrals in (11), we have

Lemma 1.
(1). If $\hat{q}$ is in $C(Lp(\Gamma):[0,T])$ with $p>2$, then the single-layer heat potential:

$$\int_0^t d\tau\int_\Gamma \hat{q}(y,\tau)\,u^*(y,\tau:x,t)\,d\Gamma\ \in C(\mathbf{R}^3\times[0,\infty))\,.$$

(2). If $v_j(x)$ is continuous in the closure of Ω, then the volume heat potential:

$$\int_0^t d\tau\int_\Omega u(y,\tau)\,v_j(y,\tau)\frac{\partial u^*}{\partial x_j}(y,\tau:x,t)\,d\Omega\ \in C(\mathbf{R}^3\times[0,\infty))$$

One of the advantages of the integral equation approach is that one can treat the continuous function even if the Neumann data $\hat{q}$ are discontinuous on the boundary.

Take a point $x\in\Gamma-\gamma$. We know the jump relation for the double-layer heat potential in the form:

$$\lim_{\substack{z \to x \\ z \in \Omega^\circ}} \int_0^t d\tau \int_\Gamma u(y,\tau)\, h^*(y,\tau : z, t')\, d\Theta_z(y)$$

$$= -\frac{1}{2\lambda} u(x,t) + \int_0^t d\tau \int_\Gamma u(y,\tau)\, h^*(y,\tau : x,t)\, d\Theta_x(y) \tag{15}$$

The integral appearing on the right hand side is defined so far only at points x on the smooth boundary. However, it can be completed to be a continuous function on the whole boundary Γ. The value at the point $\xi \in \gamma$ is given from the relation:

$$\lim_{\substack{x \to \xi \\ \xi \in \Gamma - \gamma}} \int_0^t d\tau \int_\Gamma u(y,\tau)\, h^*(y,\tau : x,t)\, d\Theta_x(y)$$

$$= -\frac{1}{2\lambda}\left(1 - \frac{\Theta(\xi)}{2\pi}\right) u(\xi,t) + \int_0^t d\tau \int_\Gamma u(y,\tau)\, h^*(y,\tau : \xi,t)\, d\Theta_\xi(y) \tag{16}$$

By combining (11), (15), and (16), we see that the unknown $u(x,t)$ at all $x \in \Omega \cup \Gamma$ is given by the solution of the following integral equation:

$$u(x,t) = \frac{1}{2}\left(2 - \frac{\Theta(x)}{2\pi}\right) u(x,t)$$

$$- \lambda \int_0^t d\tau \int_\Gamma u(y,\tau)\, h^*(y,\tau : x,t)\, d\Theta_x(y)$$

$$+ \int_0^t d\tau \int_\Omega u(y,\tau)\, v_j(y,\tau) \frac{\partial u^*}{\partial x_j}(y,\tau : x,t)\, d\Omega \tag{17}$$

$$- \int_0^t d\tau \int_\Gamma \hat{q}(y,\tau)\, u^*(y,\tau : x,t)\, d\Gamma + \int_0^t d\tau \int_\Omega \overset{\circ}{u}(y)\, u^*(y,0 : x,t)\, d\Omega(y)$$

This equation is regarded as a Volterra–Fredholm integral equation of the second kind. The equation involves not only integrals on the boundary but integrals defined on the domain.

3. Solution of the Integral Equation

We shall consider the existence of the solution of integral equation (17) in the Banach space of continuous functions $C(\bar{\Omega}\times[0,T])$ equipped with the supremum norm. To this end we shall introduce integral operators according to the following definitions:

$$Qu(x,t) := \frac{1}{2}\left(2 - \frac{\Theta(x)}{2\pi}\right)u(x,t) - \lambda\int_0^t d\tau \int_{\substack{0<|y-x|\le\delta \\ y\in\Gamma}} u(y,\tau)\,\overset{*}{h}(y,\tau:x,t)\,d\Theta_x(y)\ , \tag{18}$$

$$Vu(x,t) := -\lambda\int_0^t d\tau \int_{\substack{\delta<|y-x| \\ y\in\Gamma}} u(y,\tau)\,\overset{*}{h}(y,\tau:x,t)\,d\Theta_x(y)\ , \tag{19}$$

$$Wu(x,t) := \int_0^t d\tau \int_\Omega u(y,\tau)\,v_j(y,\tau)\frac{\partial \overset{*}{u}}{\partial x_j}(y,\tau:x,t)\,d\Omega\ , \tag{20}$$

and

$$g(x,t) := -\int_0^t d\tau \int_\Gamma \hat{q}(y,\tau)\,\overset{*}{u}(y,\tau:x,t)\,d\Gamma + \int_0^t d\tau \int_\Omega \overset{\circ}{u}(y)\,\overset{*}{u}(y,0:x,t)\,d\Omega(y)\ . \tag{21}$$

Here, $g(x,t)$ is regarded as known continuous function. The integral equation (17) is now written in the form:

$$u = (Q+V)u + Wu + g = Hu + Wu + g \tag{22}$$

Lemma 2. If Γ is the quasi-Wendland surface, then it holds that
(i). Q is a contraction in $C(\bar{\Omega}\times[0, T])$ *for some sufficiently small* δ,
(ii). V is completely continuous in $C(\bar{\Omega}\times[0, T])$ *with that* δ *as above, and*
(iii). W is completely continuous in $C(\bar{\Omega}\times[0, T])$.

Proof. For the proof of *(ii)* and *(iii)*, see Onishi[1987]. We shall show an outline of the proof of *(i)* here. First for $x \in \Omega$, we see that

$$Qu(x, t) = -\lambda \int_0^t d\tau \int_{0<|y-x|\leq\delta} u(y, \tau)\, h^*(y, \tau : x, t)\, d\Theta_x(y) .$$

It follows that

$$|Qu(x, t)|$$

$$\leq \lambda \int_{0<|y-x|\leq\delta} |d\Theta_x(y)| \int_0^t \frac{r^3}{2\lambda(t-\tau)} \left(\frac{1}{2\sqrt{\pi\lambda(t-\tau)}}\right)^3 \exp\left[-\frac{r^2}{4\lambda(t-\tau)}\right] d\tau \, \| u \|$$

with $\| u \| := \max_{\bar{\Omega}\times[0, T]} |u(x, t)|$.

To evaluate integrals we use the variable transformation:

$$\tau \mapsto \sigma = \frac{r^2}{2\sqrt{\lambda(t-\tau)}} , \qquad \text{which implies}$$

$$\sigma^2 = \frac{r^2}{4\lambda(t-\tau)} , \quad t-\tau = \frac{r^2}{4\lambda\sigma^2} , \quad d\tau = \frac{r^2}{2\lambda\sigma^3} d\sigma$$

Then we see

$$|Qu(x, t)| \leq \lambda \int_{0<|y-x|\leq\delta} |d\Theta_x(y)| \int_{\frac{r}{2\sqrt{\lambda t}}}^{\infty} 2\sigma^2 r \left(\frac{\sigma}{\sqrt{\pi}\, r}\right)^3 e^{-\sigma^2} \frac{r^2}{2\lambda\sigma^3} d\sigma \;\; \| u \|$$

$$\le \frac{1}{4\pi} \int_{0<|y-x|\le\delta} \left\{ \frac{4}{\sqrt{\pi}} \int_0^\infty \sigma^2 e^{-\sigma^2} d\sigma \right\} |d\Theta_x(y)| \, \| u \|$$

$$= \frac{1}{4\pi} \int_{0<|y-x|\le\delta} |d\Theta_x(y)| \, \| u \| \quad \text{using} \quad \int_0^\infty \sigma^2 e^{-\sigma^2} d\sigma = \sqrt{\pi} / 4 .$$

From the relation

$$\sup_{x\in\mathbf{R}^3} \int_\Gamma |d\Theta_x(y)| = A < +\infty ,$$

we can choose $\delta(q)$ so small that

$$\frac{1}{4\pi} \int_{0<|y-x|\le\delta} |d\Theta_x(y)| \le q \qquad (0<q<1) .$$

Second for $x\in\Gamma$ we see that

$$| Qu(x,t) | \le \left\{ \frac{1}{2} \left| 2 - \frac{\Theta(x)}{2\pi} \right| + \frac{1}{4\pi} \int_{0<|y-x|\le\delta} |d\Theta_x(y)| \right\} \| u \|$$

$$= \frac{1}{4\pi} \left\{ \int_{0<|y-x|\le\delta} | d\Theta_x(y) | + | 4\pi - \Theta(x) | \right\} \| u \|$$

$$= W_\delta(x) \, \| u \|$$

From the assumption in (8), we can take δ sufficiently small that $W\delta(x) \le q$ *with* $\omega<q<1$.
(q.e.d.)

Hence the operator $I-Q$ has the inverse such that

$$\| (I-Q)^{-1} \| \le 1/(1-q) \tag{23}$$

Put $u := (I-Q)^{-1}\, w$. Then the equation (22) is equivalent to

$$w = (V + W) (I - Q)^{-1} w + g =: Kw + g \tag{24}$$

Here, the operator K is also completely continuous. We consider following iterated integrals:

$$K^0 w(x, t) = w(x, t)$$

$$K^n w(x, t) = K K^{n-1} w(x, t) \quad \text{for} \quad n = 1, 2, \ldots \tag{25}$$

Then we have

Lemma 3. Operators K^n in $C(\Omega\times[0, T])$ *are bounded as*

$$\| K^n \| \leq \frac{[C_1 t^{\theta} \Gamma(\theta)]^n}{\Gamma(n\theta + 1)} \qquad (n = 0, 1, 2, \ldots)$$

with some constant $C_1 > 0$ *and* $0 < \theta < 1/2$.

The lemma can be proved in the same way as the lemma 5 in Onishi[1987]. Now we have the existence theorem:

Theorem 1. Suppose that Γ *is the quasi-Wendland surface and* $\hat{q}$ *is in* $C(Lp(\Gamma):[0,T])$ *with* $p>2$. *Then the integral equation is uniquely solvable and the operator* $I-H-W$ *in* $C(\bar{\Omega}\times[0, T])$ *is inversely stable.*

Proof. The solution w is given by the Neumann series:

$$w(x, t) = \sum_{n=0}^{\infty} K^n g (x, t)$$

The series is absolutely uniformly convergent due to the Lemma 3. The solution u is given by

$$u(x,t) = (I-Q)\sum_{n=0}^{\infty} K^n g(x,t) = (I-H-W)^{-1} g(x,t)$$

with

$$\| (I-H-W)^{-1} \| \leq (1+q) \sum_{n=0}^{\infty} \frac{[C_1 T^{\theta} \Gamma(\theta)]^n}{\Gamma(n\theta+1)}$$

4. Galerkin Approximations

We shall consider briefly the Galerkin approximation to the solution of equation (22), and show the optimal rate of uniform convergence in the space of continuous functions. To this end, let P_n (n = 1, 2, ...) be projections mapping $C(\bar{\Omega}\times[0,T])$ onto closed subspaces E_n of $C(\bar{\Omega}\times[0,T])$. We assume that P_n satisfies following two conditions:

$$\| (I-P_n)(H+W) \| \to 0 \tag{26}$$

and

$$\| (I-P_n) g \| \to 0 \tag{27}$$

as $n \to \infty$.

The Galerkin method is equivalent to finding solutions u_n of the equation:

$$P_n(u_n - Hu_n - Wu_n - g) = 0 \tag{28}$$

in E_n . As an immediate consequence of the theorem 15.3 in Krasnosel'skii *et al.*[1972], we can deduce

Theorem 2. Assume that $P_n(H+W)$ are uniformly bounded with respect to n. Then , for sufficiently large n it holds that
(i) equation (28) is uniquely solvable,
(ii) $u_n \to u$ uniformly in $C(\bar{\Omega}\times[0,T])$, and
(iii) there exist two constants c_1 , $c_2 > 0$ such that

$$c_1 \| (I-P_n)u \| < \| u_n - u \| < c_2 \| (I-P_n)u \| .$$

The last inequalities read the optimal order of convergence.

Conclusions

We presented an application of the boundary element method to the Neumann problem of heat convection conduction problem. Emphasis was put on the advantages of the integral equation method by assuming that

(i) the surface is not smooth in the sense that it consists of a finite number of quasi-Wendland subsurfaces,

(ii) the Neumann data may not be bounded in the sense that the boundary flux is a pth power summable function with $p > 2$.

$C2$-smoothness on the surface was not required. It was shown that the corresponding integral equation has a unique continuous solution at all Peclet numbers and that Galerkin approximation is stable and convergent.

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