数理解析研究所講究録
第 704 巻 1989 年 18-27

Fourier hyperfunctions of general type

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In this report, we will show the construction of the theory of Fourier hyperfunctions of general type studied by the duality method, namely by the functional analytic method. The duality method was originated by L.Schwartz first when he established the theory of distributions as a generalization of the concept of functions. Thereby, generalized functions are realized as elements of the dual space of a certain function space and vector valued ones are realized as continuous linear mappings from a certain function space into a topological vector space.

Among generalized functions, there exist Radon measures, distributions, ultradistributions, infrahyperfunctions, analytic functionals, Sato hyperfunctions, Fourier hyperfunctions, modified Fourier hyperfunctions, mixed Fourier hyperfunctions, partial Fourier hyperfunctions, partial modified Fourier hyperfunctions, partial mixed Fourier hyperfunctions and those ones in vector valued case. Especially, the space of each type of Fourier hyperfunctions and vector valued ones has the characteristic property that it is stable under Fourier transformation when the ground space whose variables are the ones of Fourier hyperfunction part is the whole space.

These types of Sato-Fourier hyperfunctions can be obtained as specializations of our Fourier hyperfunctions of general type and are distinguished from each others by the difference of topologies of their ground spaces. In this sense, the theory of Fourier hyperfunctions of general type is the unified theory of those types of Sato-Fourier hyperfunctions.

We construct the sheaf of Fourier hyperfunctions of general type by using the generalization of Schapira-Junker Theorem and prove that this sheaf is flabby. This generalization of the Schapira-Junker Theorem gives the most general method of constructing flabby sheaves. The Schapira-Junker Theorem is the following

Theorem 1 (Schapira-Junker). Let X be a σ-compact locally compact topological space satisfying the second axiom of countability. We assume that, for every compact subset K of X, there exists a Fréchet space F_K such that the following conditions are fulfilled:

(1) For two compact subsets K_1, K_2 of X with $K_1 \subset K_2$, there exists a continuous injection $i_{K_1,K_2}: F_{K_1} \longrightarrow F_{K_2}$.

(2) If K_1, K_2 are two compact subsets of X with $K_1 \subset K_2$ and each connected component of K_2 does meet K_1, then i_{K_1,K_2} has a dense image.

(3) For two compact subsets K_1 and K_2 of X and for every $u \in F_{K_1 \cup K_2}$, there exist $u_1 \in F_{K_1}$ and $u_2 \in F_{K_2}$ so that $u = u_1 + u_2$ holds, where u_1 and u_2 are considered as elements of $F_{K_1 \cup K_2}$ by virtue of (1).

(4) For every at most countable family $\{K_i; i\in I\}$ of compact subsets of X, $\bigcap_{i\in I} F_{K_i} = F_K$ holds, where $K = \bigcap_{i\in I} K_i$.

(5) $F_\phi = 0$.

Then there exists one and only one flabby sheaf $\mathcal{F}$ over X so that, for every compact subset K of X, $\Gamma_K(X,\mathcal{F}) = F_K$ holds.

Using this theorem, we construct the sheaf of Fourier hyperfunctions of general type in the following way.

For a natural number n, we set $\tilde{\mathbf{C}}^{\mathbf{n}} = \mathbf{D}^{\mathbf{n}} \times i\mathbf{R}^{\mathbf{n}}$ and $\mathbf{E}^{\mathbf{n}} = \mathbf{D}^{\mathbf{2n}}$. Here $\mathbf{D}^{\mathbf{n}}$ denotes the radial compactification of $\mathbf{R}^{\mathbf{n}}$ in the sense of Kawai and $\mathbf{E}^{\mathbf{n}} = \mathbf{D}^{\mathbf{2n}}$ denotes the radial compactification of $\mathbf{C}^{\mathbf{n}}$ identified with $\mathbf{R}^{\mathbf{2n}}$. Then, for a 3-tuple $n=(n_1,n_2,n_3)$ of nonnegative integers with $|n| = n_1+n_2+n_3 \neq 0$, we put $\mathbf{K}^{\mathbf{n}} = \mathbf{C}^{\mathbf{n_1}} \times \tilde{\mathbf{C}}^{\mathbf{n_2}} \times \mathbf{E}^{\mathbf{n_3}}$ and $\tilde{\tilde{\mathbf{D}}}^{\mathbf{n}} = \mathbf{R}^{\mathbf{n_1}} \times \mathbf{D}^{\mathbf{n_2}} \times \mathbf{D}^{\mathbf{n_3}}$.

Then we define the sheaf $\mathcal{O}_*$ to be the sheafification of the presheaf $\{\mathcal{O}_*(\Omega); \Omega \subset \mathbf{K}^{\mathbf{n}}, \text{ open}\}$, where the section module $\mathcal{O}_*(\Omega)$ on an open set Ω in $\mathbf{K}^{\mathbf{n}}$ is the space of all holomorphic functions f on $\Omega \cap \mathbf{C}^{|\mathbf{n}|}$ such that, for any compact set K in Ω, there exists some positive constant δ so that they satisfy the condition

$$\sup\{|f(z)|\exp(\delta|z|); z \in K \cap \mathbf{C}^{|\mathbf{n}|}\} < \infty.$$

If K is a compact set K in $\mathbf{K}^{\mathbf{n}}$, then we endow $\mathcal{O}_*(K)$ with the inductive limit topology $\varinjlim_m \mathcal{O}_c^m(U_m)$, where $\{U_m\}$ is a fundamental system of neighborhoods of K satisfying $U_m \supset\supset U_{m+1}$ and $\mathcal{O}_c^m(U_m)$ is the Banach space of all functions f(z) which are holomorphic on $U_m \cap \mathbf{C}^{|\mathbf{n}|}$ and continuous on $U_m^a \cap \mathbf{C}^{|\mathbf{n}|}$ and satisfy the condition $|f(z)| \leq C\exp(-|z|/m)$, $z \in U_m^a \cap \mathbf{C}^{|\mathbf{n}|}$, for some positive constant C.

Then $\mathcal{O}_*(K)$ is a nuclear DFS space.

Now, in this report we assume that E is a Fréchet space. We put $\mathcal{A}_*(K)=\mathcal{O}_*(K)$ and $\mathcal{A}_*^!(K;E)=L(\mathcal{A}_*(K);E)$ for any compact set K in $\tilde{\tilde{D}}^n$ and call the elements of $\mathcal{A}_*^!(K;E)$ Fourier analytic linear mappings of general type. Then we have $\mathcal{A}_*^!(K;E)\cong \mathcal{A}_*(K)'\hat{\otimes}E$.

Then we have the following

Theorem 2. For every compact subset K of $\tilde{\tilde{D}}^n$ we assign the space $\mathcal{A}_*^!(K;E)$ of Fourier analytic linear mappings of general type on K. Then the following are valid.

(1) If K_1 and K_2 are two compact subsets of $\tilde{\tilde{D}}^n$ with $K_1\subset K_2$, there exists a continuous injection $i_{K_1,K_2}:\mathcal{A}_*^!(K_1;E)\longrightarrow\mathcal{A}_*^!(K_2;E)$.

(2) If K_1 and K_2 are as in (1) and each connected component of K_2 does meet K_1, then i_{K_1,K_2} has a dense image.

(3) If K_1 and K_2 are two compact subsets of $\tilde{\tilde{D}}^n$, then, for every $u\in\mathcal{A}_*^!(K_1\cup K_2;E)$, there exist $u_1\in\mathcal{A}_*^!(K_1;E)$ and $u_2\in\mathcal{A}_*^!(K_2;E)$ so that $u=u_1+u_2$ holds, where u_1 and u_2 are considered as elements of $\mathcal{A}_*^!(K_1\cup K_2;E)$ by virtue of (1).

(4) For every at most countable family $\{K_i\}_{i\in I}$ of compact subsets of $\tilde{\tilde{D}}^n$, $\bigcap_{i\in I}\mathcal{A}_*^!(K_i;E)=\mathcal{A}_*^!(\bigcap_{i\in I}K_i;E)$ holds.

(5) $\mathcal{A}_*^!(\phi;E)=0$.

Then, by virtue of Theorem 1, we have the following

Theorem 3. There exists one snd only one flabby sheaf $\mathcal{M}$ over $\tilde{\tilde{D}}^n$ so that, for every compact subset K of $\tilde{\tilde{D}}^n$,

$\Gamma_K(\tilde{\tilde{\mathbf{D}}}^{\mathbf{n}},\mathcal{M})=\mathcal{A}_*'(K;E)$ holds.

Definition. The sheaf $\mathcal{M}$ is said to be the sheaf of Fourier hyperfunctions of general type over $\tilde{\tilde{\mathbf{D}}}^{\mathbf{n}}$ and a section f of $\mathcal{M}$ on an open set Ω in $\tilde{\tilde{\mathbf{D}}}^{\mathbf{n}}$ is said to be a Fourier hyperfunction of general type on Ω.

By the specialization of $n=(n_1,n_2,n_3)$, we have the sheaves (1) $\mathcal{A}$, (2) $\underset{\sim}{\mathcal{A}}$, (3) $\underset{\approx}{\mathcal{A}}$, (4) $\mathcal{A}_{\#}$, (5) $\mathcal{A}_{b}$, (6) $\mathcal{A}_{\flat}$ and (7) $\mathcal{A}_*$ for the cases (1) $n=(n_1,0,0)$, (2) $n=(0,n_2,0)$, (3) $n=(0,0,n_3)$, (4) $n=(0,n')=(0,n_2,n_3)$, (5) $n=(n_1,n_2,0)$, (6) $n=(n_1,0,n_2)$ and (7) $n=(n_1,n_2,n_3)$ respectively. We denote by $\mathcal{A}_\alpha$ any one of these sheaves. We consider the sheaves $\mathcal{A}_\alpha$ on M^m and $\mathcal{A}_\beta$ on M^n. Here M^m and M^n are the corresponding ground spaces. Then we have the following isomorphisms

$$\mathcal{A}_\alpha(K')'\hat{\otimes}\mathcal{A}_\beta(K'')'\cong L(\mathcal{A}_\alpha(K');\mathcal{A}_\beta(K'')')\cong\mathcal{A}_{\alpha\beta}(K'\times K'')',$$

for $K'\subset\subset M^m$ and $K''\subset\subset M^n$ (compacts) and the following isomorphisms

$$\mathcal{A}_\alpha'(K';E_1)\hat{\otimes}_\omega\mathcal{A}_\beta'(K'';E_2)\cong\mathcal{A}_{\alpha\beta}'(K'\times K'';E).$$

Here E_1 and E_2 are Fréchet spaces and we put $E=E_1\hat{\otimes}_\omega E_2$ and ω stands for ε- or π-topology and $\mathcal{A}_{\alpha\beta}=\mathcal{A}_\alpha\otimes\mathcal{A}_\beta$ is the direct tensor product of sheaves $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$.

Now we define the convolution of $u\in\mathcal{A}_\alpha'(M^n;E_1)$ and $v\in\mathcal{A}_\alpha'(M^n;E_2)$, one of which has a compact support in $\mathbf{R}^{|\mathbf{n}|}$, in the following way,

$$(u*_\omega v)(f(x))=(u_\xi\otimes_\omega v_\eta)(f(\xi+\eta)),\ \text{for}\ f\in\mathcal{A}_\alpha(M^n).$$

Now we will define the differentiation of $u\in\mathcal{A}_\alpha'(K;E)$ by the following way:

$$(\partial u/\partial x_j)(f)=-u(\partial f/\partial x_j),\ (j=1,2,---,|n|),$$

for $f \in \mathcal{A}_\alpha(K)$.

Then we have

$$\partial u/\partial x_j = \lim_{h_j \to 0} \frac{1}{h_j}(\tau_{-h}u - u).$$

Here we put $e_j = (\delta_{jk})_{1 \leq k \leq |n|}$, and $h = h_j e_j$ and we define the translation operator τ_h by the formula

$$\tau_h f(x) = f(x-h), \text{ for } f \in \mathcal{A}_\alpha(K),$$

$$\tau_h u(f) = u(\tau_{-h} f), \text{ for } u \in \mathcal{A}'_\alpha(K;E) \text{ and } f \in \mathcal{A}_\alpha(K).$$

For the integration, we have the following

<u>Theorem 4</u>. Let K be a compact set in M^n of the type $K_1 \times \cdots \times K_{n_1} \times \mathbf{D}^{n_2} \times \mathbf{D}^{n_3}$ with $K_j = [-a_j, a_j]$, $(a_j > 0)$, $(j = 1, \cdots, n_1)$. Take the natural projection $F(x_1, \cdots, x_{|n|}) = (x_1, \cdots, \hat{x}_j, \cdots, x_{|n|})$ for $x = (x_1, \cdots, x_{|n|}) \in K \cap \mathbf{R}^{|n|}$. Here $\hat{x}_j$ denotes the omission of x_j. Then, for any $v \in {}_{|n|}\mathcal{A}'_\alpha(K;E)$, there exists $u \in {}_{|n|}\mathcal{A}'_\alpha(K;E)$ such that $\partial u/\partial x_j = v$. Such two solutions u_1 and u_2 are different one another by an arbitrary analytic linear mapping in ${}_{|n|-1}\mathcal{A}'_\alpha(F(K);E)$.

Similarly, we can define canonically those corresponding operations for Fourier hyperfunctions of general type.

By the above specialization of $n = (n_1, n_2, n_3)$, we have the sheaves of Sato-Fourier hyperfunctions (1) ${}^{E_i}\mathcal{B}$, (2) ${}^{E_i}\mathcal{R}$, (3) ${}^{E_i}\mathcal{Q}$, (4) ${}^{E_i}\mathcal{P}$, (5) ${}^{E_i}(\mathcal{BR})$, (6) ${}^{E_i}(\mathcal{BQ})$ and (7) $\mathcal{M}^i = {}^{E_i}(\mathcal{BP})$, i=1,2 as the specializations of $\mathcal{M}$, where E_1 and E_2 are Fréchet spaces and we put $E = E_1 \hat{\otimes}_\omega E_2$. Let both ${}^{E_1}\mathcal{F}$ and ${}^{E_2}\mathcal{G}$ denote one of the above sheaves of Sato-Fourier hyperfunctions over M^m and M^n respectively. Then we have the following

Theorem 5. In the above notations, we have the isomorphism $^{E}(\mathcal{F}\mathcal{O}) \cong {}^{E_1}\mathcal{F}\hat{\otimes}_{\omega}{}^{E_2}\mathcal{O}$. Namely, for any open sets Ω_1 in M^m and Ω_2 in M^n, we have by definition

$$(\mathcal{F}\hat{\otimes}\mathcal{O})(\Omega_1\times\Omega_2;E) \cong \mathcal{F}(\Omega_1;E_1)\hat{\otimes}_{\omega}\mathcal{O}(\Omega_2;E_2).$$

This is an analog of Schwartz' Kernel Theorem.

Now we consider the convolution. If M^n is compact, $^{E_2}\mathcal{F}(M^n) \equiv \mathcal{F}(M^n;E_2)=\mathcal{A}'_{\alpha}(M^n;E_2)$. Thus, in this case, we have nothing special to do with convolution products of vector valued Fourier hyperfunctions of each type.

If M^n is not compact, let $T \in \mathcal{F}(M^n;E_2)$ and $u \in \mathcal{A}'(R^{|n|};E_1)$. We define $u*T$ as follows. Let Ω_j be an open ball with center at the origin and with radius j, and

$$\overline{T}_j \in \mathcal{A}'_{\alpha}(\Omega_j^a;E_2),\quad \overline{T}_j|\Omega_j=T|\Omega_j.$$

Here $\overline{T}_j|\Omega_j$ denotes the image of the canonical mapping $\mathcal{A}'_{\alpha}(\Omega_j^a;E_2)\longrightarrow\mathcal{F}(\Omega_j;E_2)$. Let k be chosen so that $u \in \mathcal{A}'(\Omega_k;E_1)$. If $j' \geqq j > k$, we have

$$(u*_{\omega}\overline{T}_j)|\Omega_{j-k}=(u*_{\omega}\overline{T}_{j'})|\Omega_{j-k}.$$

The sequence $(u*_{\omega}\overline{T}_j)|\Omega_{j-k}$ defines a vector valued Sato-Fourier hyperfunction which we denote by $u*_{\omega}T$.

Now let E be a Fréchet space and consider M^n and $^{E}\mathcal{F}$ as in the above. Let Ω be an open set in M^n. If Ω is relatively compact and $T \in \mathcal{F}(\Omega;E)$, let $\overline{T}\in\mathcal{A}'_{\alpha}(\Omega^a;E)$ such that $\overline{T}|\Omega=T$. Then we define $\partial T/\partial x_j$ as follows:

$$(\partial\overline{T}/\partial x_j)|\Omega=\partial T/\partial x_j.$$

If Ω is not necessarily relatively compact, we take an arbitrary open covering $\{\Omega_j\}_{j=0}^{\infty}$ of Ω with $\Omega_j \subset\subset \Omega_{j+1}$. Then, for any

$T \in \mathcal{F}(\Omega;E)$, let $\overline{T}_j \in \mathcal{A}'_\alpha(\overline{\Omega}_j;E)$ such that

$\overline{T}_j|\Omega_j = T|\Omega_j \equiv T_j$.

Then we have

$(\partial\overline{T}_{j+k}/\partial x_h)|\Omega_j = (\partial\overline{T}_j/\partial x_h)|\Omega_j$, for $k \geq 0$.

Hence $(\partial\overline{T}_j/\partial x_h)|\Omega_j$'s define a vector valued Sato Fourier hyperfunction which depends only on T and by which we define $\partial T/\partial x_h$. That is, we have

$(\partial\overline{T}_j/\partial x_h)|\Omega_j = (\partial T/\partial x_h)|\Omega_j$.

As for the integration, we have the following

Theorem 6. We use the above notations. Let Ω be an open set in M^n. Then, for any $T \in \mathcal{F}(\Omega;E)$, there exists $S \in \mathcal{F}(\Omega;E)$ such that $\partial S/\partial x_j = T$. Such two solutions S_1 and S_2 are different one another by an arbitrary vector valued Sato-Fourier hyperfunction in $F^{-1}({}^E\mathcal{F}|M^{n-e})(\Omega)$, where F denotes the natural projection such as $F(x_1,---,x_{|n|}) = (x_1,---,\hat{x}_j,---,x_{|n|})$ for $x=(x_1,---,x_{|n|}) \in \mathbf{R}^{|n|}$ and M^{n-e} denotes the closure of $\mathbf{R}^{|n|-1} = F(\mathbf{R}^{|n|})$ in M^n and $F^{-1}({}^E\mathcal{F}|M^{n-e})$ denotes the inverse image of the restriction of ${}^E\mathcal{F}$ to M^{n-e}.

At last we will define the Fourier transformation of Fourier hyperfunctions of general type. We will concern only with those variables which are related to the Fourier hyperfunction parts. So that we have only to treat the vector valued mixed Fourier hyperfunctions. We will first define the Fourier transformation of test functions as follows:

(I) for $\varphi \in \mathcal{A}(\mathbf{D}^n)$,

$$(\mathcal{F}_1\varphi)(\xi) = \int_{\mathbf{R}^n} \exp(i\langle x,\xi\rangle)\varphi(x)dx,$$

$<x,\xi>=x_1\xi_1+\cdots+x_n\xi_n$,

(II) for $\varphi\in\underset{\approx}{\mathcal{A}}(D^n)$,

$$(\mathcal{F}_2\varphi)(\xi)=\int_{R^n}\exp(i<x,\xi>)\varphi(x)dx,$$

(III) for $\varphi\in\mathcal{A}_{\#}(D^n)$, $n=(n_1,n_2)$,

$$(\mathcal{F}\varphi)(\xi)=\int_{R^{|n|}}\exp(i<x,\xi>)\varphi(x)dx.$$

Then we have $\mathcal{F}=\mathcal{F}_1\hat{\otimes}\mathcal{F}_2$. Then $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}$ are topological automorphism of $\underset{\sim}{\mathcal{A}}(D^n)$, $\underset{\approx}{\mathcal{A}}(D^n)$ and $\mathcal{A}_{\#}(D^n)$ respectively.

(IV) for $T\in\mathcal{J}(D^n;E)=\mathcal{A}'_{\#}(D^n;E)$, we define $\mathcal{F}^*T$ by the formula

$$(\mathcal{F}^*T)(\varphi)=T(\mathcal{F}\varphi),\text{ for any }\varphi\in\mathcal{A}_{\#}(D^n).$$

Then we have the following

<u>Theorem 7(Paley-Wiener theorem)</u>. Let Γ be a closed and strictly convex cone in $R^{|n|}$ and K its closure in D^n. For the sake of simplicity we assume that the vertex of the cone Γ be at the origin. Let $T\in\mathcal{J}(D^n;E)$. Then $T\in\mathcal{A}'_{\#}(K;E)$ if and only if

$$J(\zeta)=T_z(\exp(i<z,\zeta>))\in\mathcal{O}^{\#}(\text{int}[R^{|n|}\times i\Gamma^\circ]^a;E)$$

holds. Here we put $<z,\zeta>=z_1\zeta_1+\cdots+z_{|n|}\zeta_{|n|}$ and $\Gamma^\circ=\{\xi\in R^{|n|};$ $<x,\xi>\geq 0$ for any $x\in\Gamma\}$ is the polar set of Γ and $\mathcal{O}^{\#}(\text{int}[R^{|n|}\times i\Gamma^\circ]^a;E)$ denotes the space of E-valued slowly increasing holomorphic functions on $\text{int}[R^{|n|}\times i\Gamma^\circ]^a$.

In the above theorem, the space $\mathcal{O}^{\#}(\Omega)$ on an open set Ω in $F^n=\tilde{C}^{n_1}\times E^{n_2}$ is defined as follows: $\mathcal{O}^{\#}(\Omega)$ is the space of all holomorphic functions $f(z)$ on $\Omega\cap C^{|n|}$ such that,for any $\varepsilon>0$ and any compact set K in Ω, the estimate $\sup\{|f(z)|\exp(-\varepsilon|z|);z\in K\cap C^{|n|}\}<\infty$ holds.

Reference

[1] Y. Ito, Fourier hyperfunctions of general type, to appear in J. Math. Kyoto Univ..