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The Fekete-Szegö problem for strongly close-to-convex functions.

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INTRODUCTION

Denote by S the class of normalized analytic univalent functions f defined for $z \in D = \{z : |z| < 1\}$ by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

A classical theorem of Fekete and Szegö [2] states that for $f \in S$ given by (1),

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 0 \\ 1 + 2e^{-2\mu/(1-\mu)}, & \text{if } 0 \leq \mu < 1 \\ 4 - 3\mu, & \text{if } \mu \geq 1. \end{cases}$$

This inequality is sharp in the sense that for each μ there exists a function in S such that equality holds. Recently Pfluger [8] has considered the problem when μ is complex. In the case of C, S^* and K, the subclasses of convex, starlike and close-to-convex functions respectively, the above inequalities can be improved [5,6]. In particular for $f \in K$ and given by (1), Keogh and Merkes [5] showed that

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 1/3, \\ 1/3 + 4/9\mu, & \text{if } 1/3 \leq \mu \leq 2/3, \\ 1, & \text{if } 2/3 \leq \mu \leq 1, \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases}$$

Again, for each μ, there is a function in K such that equality holds. In this paper we extend this result to the class $K(\beta)$ of strongly close-to-convex functions of order β in the sense of Pommerenke [9]. Thus

$f \in K(\beta)$ if, and only if, f, given by (1), is analytic in D and is such that there exists $g \in S^*$ satisfying

$$\left|\arg \frac{zf'(z)}{g(z)}\right| \leq \frac{\pi\beta}{2}, \tag{2}$$

for $z \in D$ and $\beta \geq 0$. Clearly $K(0) = C$, $K(1) = K$ and when $0 \leq \beta \leq 1$, $K(\beta)$ is a subset of K and hence contains only univalent functions. However in [4], Goodman showed that $K(\beta)$ can contain functions with unbounded valence for $\beta > 1$.

Recently, Koepf [7] has considered the Fekete-Szegö problem for $K(\beta)$ and obtained sharp results for some particular values of μ, all of which, with the exception of the case $\mu = 1$ and $\beta \geq 1$, are contained in the following result.

RESULTS

THEOREM. *Let $f \in K(\beta)$ and be given by (1), then for $0 \leq \beta \leq 1$,*

$$\begin{aligned}
|a_3 - \mu a_2^2| &\leq 1 - \mu + \frac{\beta(2-3\mu)(\beta+2)}{3}, && \text{if } \mu \leq \frac{2\beta}{3(\beta+1)},\\
&\leq 1 - \mu + \frac{2\beta}{3} + \frac{\beta(2-3\mu)^2}{3[2-\beta(2-3\mu)]}, && \text{if } \frac{2\beta}{3(\beta+1)} \leq \mu \leq \frac{2}{3},\\
&\leq \frac{2\beta+1}{3}, && \text{if } \frac{2}{3} \leq \mu \leq \frac{2(\beta+2)}{3(\beta+1)},\\
&\leq \mu - 1 + \frac{\beta(3\mu-2)(\beta+2)}{3}, && \text{if } \mu \geq \frac{2(\beta+2)}{3(\beta+1)},
\end{aligned}$$

and for $\beta > 1$, the first two inequalities hold. For each μ, there is a function in $K(\beta)$ such that equality holds.

We shall require the following:

LEMMA 1 ([**10**], p. 166). *Let $h \in P$, i.e., let h be analytic in D and satisfy $\operatorname{Re} h(z) > 0$ for $z \in D$, with $h(z) = 1 + c_1 z + c_2 z^2 + \ldots$, then*

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1^2|}{2}.$$

LEMMA 2 ([6], Lemma 3). *Let $g \in S^*$ with $g(z) = z + b_2z^2 + b_3z^3 + \ldots$, then for μ real,*

$$|b_3 - \mu b_2^2| \leq \max\{1, |3 - 4\mu|\}.$$

We note that Lemma 2 above can easily be extended to the wider class $S^*(\alpha)$ of strongly starlike functions of order $\alpha \geq 0$, i.e., g analytic and normalized in D and satisfying

$$\left|\arg \frac{zg'(z)}{g(z)}\right| \leq \frac{\alpha\pi}{2},$$

see e.g. [1]. In this case, one obtains the sharp inequality

$$|b_3 - \mu b_2^2| \leq \max\{\alpha, \alpha^2|3 - 4\mu|\},$$

for μ real.

PROOF OF THEOREM: It follows from (2) that we can write

$$zf'(z) = g(z)h(z)^\beta \tag{3}$$

for $g \in S^*$ and $h \in P$. Equating coefficients in (3) we obtain

$$2a_2 = \beta c_1 + b_2$$

and

$$3a_3 = \frac{\beta(\beta-1)}{2}c_1^2 + \beta c_2 + \beta c_1 b_2 + b_3,$$

so that

$$a_3 - \mu a_2^2 = \frac{1}{3}\left(b_3 - \frac{3}{4}\mu b_2^2\right) + \frac{\beta}{3}\left(c_2 + \left(\frac{\beta(2-3\mu)}{4} - \frac{1}{2}\right)c_1^2\right) + \beta\left(\frac{1}{3} - \frac{\mu}{2}\right)c_1 b_2. \tag{4}$$

We consider first the case $\frac{2\beta}{3(\beta+1)} \leq \mu \leq \frac{2}{3}$. Equation (4) gives

$$
\begin{aligned}
|a_3 - \mu a_2^2| &\le \frac{1}{3}\left|b_3 - \frac{3}{4}\mu b_2^2\right| + \frac{\beta}{3}\left|c_2 - \frac{1}{2}c_1^2\right| + \frac{\beta^2(2-3\mu)}{12}|c_1^2| \\
&\quad + \beta\left(\frac{1}{3} - \frac{\mu}{2}\right)|c_1||b_2|, \\
&\le 1 - \mu + \frac{\beta}{3}\left(2 - \frac{1}{2}|c_1^2|\right) + \frac{\beta^2(2-3\mu)}{12}|c_1^2| \\
&\quad + \frac{\beta(2-3\mu)}{3}|c_1|, \\
&= \Phi(x) \text{ say, with } x = |c_1|,
\end{aligned}
$$

where we have used Lemmas 1 and 2 and the fact that $|b_2| \le 2$ for $g \in S^*$. An elementary argument shows that the function Φ attains a maximum at $x_0 = 2(2-3\mu)/(2-\beta(2-3\mu))$, and so

$$
|a_3 - \mu a_2^2| \le \Phi(x_0),
$$

which proves the Theorem if $\mu \le 2/3$ and $\beta \ge 0$. Choosing

$$
c_1 = \frac{2(2-3\mu)}{2-\beta(2-3\mu)}, \;\; c_2 = 2, \;\; b_2 = 2 \text{ and } b_3 = 3,
$$

in (4) shows that the result is sharp. We note that $|c_1| \le 2$, i.e., $\mu \ge 2\beta/(3(\beta+1))$.

Next consider the case $\mu \le \dfrac{2\beta}{3(\beta+1)}$. Since $K(0) = C$, we may assume that $\beta > 0$. Again (4) gives

$$
\begin{aligned}
|a_3 - \mu a_2^2| &\le \frac{3\mu(\beta+1)}{2\beta}\left|a_3 - \frac{2\beta}{3(\beta+1)}a_2^2\right| + \left(1 - \frac{3\mu(\beta+1)}{2\beta}\right)|a_3|, \\
&\le \frac{3\mu(\beta+1)}{2\beta}\left(1 + \frac{2\beta}{3}\right) \\
&\quad + \left(1 - \frac{3\mu(\beta+1)}{2\beta}\right)\left(\frac{2\beta(\beta+2)}{3} + 1\right), \\
&= 1 - \mu + \frac{\beta(2-3\mu)(\beta+2)}{3},
\end{aligned}
$$

for $\beta > 0$, where we have used the result already proved in the case $\mu = 2\beta/3(\beta+1)$, and the fact that for $f \in K(\beta)$, the inequality $|a_3| \leq 1 + 2\beta(\beta+2)/3$ holds [3]. Equality is attained on choosing $c_1 = c_2 = b_2 = 2$ and $b_3 = 3$.

Suppose now that $\dfrac{2}{3} \leq \mu \leq \dfrac{2(\beta+2)}{3(\beta+1)}$. Since $g \in S^*$ we can write $zg'(z) = g(z)p(z)$ for $p \in P$, with $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$, and so equating coefficients we have that $b_2 = p_1$ and $2b_3 = p_1^2 + p_2$.

We deal first with the case $\mu = 2(\beta+2)/(3(\beta+1))$. Thus (4) gives

$$a_3 - \frac{2(\beta+2)}{3(\beta+1)}a_2^2 = \frac{1}{6}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{\beta}{3}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{\beta-1}{12(\beta+1)}p_1^2 - \frac{\beta^2 c_1^2}{6(\beta+1)} - \frac{\beta p_1 c_1}{3(\beta+1)},$$

and so if $\beta \leq 1$,

$$\begin{aligned}\left|a_3 - \frac{2(\beta+2)}{3(\beta+1)}a_2^2\right| &\leq \frac{1}{6}\left|p_2 - \frac{p_1^2}{2}\right| + \frac{\beta}{3}\left|c_2 - \frac{c_1^2}{2}\right| + \frac{(1-\beta)}{12(\beta+1)}|p_1^2| \\ &\quad + \frac{\beta^2|c_1^2|}{6(\beta+1)} + \frac{\beta|p_1 c_1|}{3(\beta+1)}, \\ &\leq \frac{1}{6}\left(2 - \frac{|p_1^2|}{2}\right) + \frac{\beta}{3}\left(2 - \frac{|c_1^2|}{2}\right) + \frac{1-\beta}{12(\beta+1)}|p_1^2| \\ &\quad + \frac{\beta^2|c_1^2|}{6(\beta+1)} + \frac{\beta|p_1 c_1|}{3(\beta+1)}, \\ &= \frac{2\beta+1}{3} - \frac{\beta}{6(\beta+1)}\left(|c_1| - |p_1|\right)^2, \\ &\leq \frac{2\beta+1}{3},\end{aligned}$$

where we have used Lemma 1.

Now write

$$\begin{aligned}a_3 - \mu a_2^2 &= \frac{(\beta+1)(3\mu-2)}{2}\left(a_3 - \frac{2(\beta+2)}{3(\beta+1)}a_2^2\right) \\ &\quad + \frac{3(\beta+1)}{2}\left(\frac{2(\beta+2)}{3(\beta+1)} - \mu\right)\left(a_3 - \frac{2}{3}a_2^2\right),\end{aligned}$$

and the result follows at once on using the Theorem already proved in the cases $\mu = 2/3$ and $\mu = 2(\beta + 2)/(3(\beta + 1))$ for $\beta \leq 1$. Equality is attained when f is given by

$$f'(z) = \frac{(1+z^2)^\beta}{(1-z^2)^{\beta+1}}.$$

We finally assume that $\mu \geq \dfrac{2(\beta+2)}{3(\beta+1)}$. Write

$$a_3 - \mu a_2^2 = \left(a_3 - \frac{2(\beta+2)}{3(\beta+1)} a_2^2\right) + \left(\frac{2(\beta+2)}{3(\beta+1)} - \mu\right) a_2^2,$$

and the result follows at once on using the Theorem already proved for $\mu = 2(\beta+2)/3(\beta+1))$ in the case $\beta \leq 1$ and the inequality $|a_2| \leq \beta+1$, which was proved in [**3**]. Equality is attained in this last case on choosing $c_1 = c_2 = b_2 = 2$ and $b_3 = 3$ in (4).

We remark that the methods used in [**5**] and [**6**], together with equation (4), suggest that in order to obtain sharp results for $\beta > 1$ and $\mu > 2/3$, an extension to the "area principle" may be required. Since $K(\beta)$ contains functions of unbounded valence for $\beta > 1$ establishing sharp estimates in this case may require deeper methods.

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