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CERTAIN CLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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1. Introduction.

Let A denote the class of function $f(z)$ analytic in the unit disk $E = \{z: |z| < 1\}$. Let V denote the subclass of A consisting functions normalized by $f(0) = 0$ and $f'(0) = 1$. The Hardamard product $(f*g)(z)$ of two functions $f(z) = \sum_{m=0}^{\infty} a_m z^m$ and $g(z) = \sum_{m=0}^{\infty} b_m z^m$ in A is given by $(f*g)(z) = \sum_{m=0}^{\infty} a_m b_m z^m$.

Let $D^{\gamma} f(z) = \dfrac{z}{(1-z)^{\gamma+1}} * f(z)$ $(\gamma > -1)$. Ruscheweyh [4] observed that $D^n f(z) = z(z^{n-1} f(z))^{(n)}/n!$ when $n \in N \cup \{0\}$, where $N = \{1, 2, 3, \ldots\}$. This symbol $D^n f(z)$, was called the nth Ruscheweyh derivative of $f(z)$ by Al-Amiri [1].

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Let T be the subclass of V consisting functions of the form $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geq 0$. Functions of this type have been studied by Silverman [5]. Let $S(n, \lambda, A, B)$ denote the class of functions $f \in T$ such that

$$(1-\lambda)\frac{D^n f(z)}{z} + \lambda \frac{D^{n+1} f(z)}{z} \prec \frac{1+Az}{1+Bz},$$

for $z \in E$, where $\lambda \geq 0$, $-1 \leq A < B \leq 1$ and $n \in N \cup \{0\}$. The class $S(0, \lambda, 2\alpha-1, 1)$ with $0 \leq \alpha < 1$ has been considered by Bhoosnurmath and Swamy [2].

In this paper, we find the coefficients inequalities and determine the extreme points, radii of starlikeness and convexity. We prove distortion theorems. We also consider the modified Hadamard product of functions in $S(n, \lambda, A, B)$. Some results obtained by Bhoosnurmath and Swamy [2] can be reduced from the corresponding results for the class $S(n, \lambda, A, B)$ by taking $n = 0$, $B \equiv 1$ and $A = 2\alpha-1$, $0 \leq \alpha < 1$.

2. Coefficients Inequality.

Theorem 1. Let $f \in T$. Then $f \in S(n, \lambda, A, B)$ if and only if

$$\sum_{m=2}^{\infty} \frac{(m+n-1)![n+1+\lambda(m-1)]}{(n+1)!(m-1)!} a_m \leq \frac{B-A}{1+B} . \tag{2.1}$$

Proof. Suppose $f \in S(n, \lambda, A, B)$. Then

$$h(z) = (1-\lambda)\frac{D^n f(z)}{z} + \lambda\frac{D^{n+1} f(z)}{z} = \frac{1+Aw(z)}{1+Bw(z)},$$

$-1 \leq A < B \leq 1$, $z \in E$, $w \in H = \{w$ analytic, $w(0) = 0$ and $|w(z)| < 1$, $z \in E\}$. From this we get

$$w(z) = \frac{1-h(z)}{Bh(z) - A} .$$

Since

$$D^n f(z) = z(z^{n-1} f(z))^{(n)}/n!$$

$$= z - \sum_{m=2}^{\infty} \frac{(m+n-1)!}{n!(m-1)!} a_m z^m ,$$

therefore

$$h(z) = 1 - \sum_{m=2}^{\infty} \frac{(m+n-1)![n+1+\lambda(m-1)]}{(n+1)!(m-1)!} a_m z^{m-1}$$

and $|w(z)| < 1$ implies

$$(2.2) \qquad \left| \frac{\sum_{m=2}^{\infty} \frac{(m+n-1)![n+1+\lambda(m-1)]}{(n+1)!(m-1)!} a_m z^{m-1}}{(B-A)-B\sum_{m=2}^{\infty} \frac{(m+n-1)![n+1+\lambda(m-1)]}{(n+1)!(m-1)!} a_m z^{m-1}} \right| < 1 .$$

Hence

$$(2.3) \qquad \operatorname{Re}\left(\frac{\sum_{m=2}^{\infty} \frac{(m+n-1)![n+1+\lambda(m-1)]}{(n+1)!(m-1)!} a_m z^{m-1}}{(B-A)-B\sum_{m=2}^{\infty} \frac{(m+n-1)![n+1+\lambda(m-1)]}{(n+1)!(m-1)!} a_m z^{m-1}} \right) < 1 .$$

We consider real values of z and take $0 \le r < 1$. Then, for $r = 0$, denominator of (2.3) is positive and so it is positive for all r with $0 \leq r < 1$, since $w(z)$ is analytic for $|z| < 1$. Then (2.3) gives

$$\sum_{m=2}^{\infty} \frac{(m+n-1)![n+1+\lambda(m-1)]}{(n+1)!(m-1)!} a_m r^{m-1} < \frac{B-A}{1+B} . \tag{2.4}$$

Letting $r \longrightarrow 1$, we get (2.1).

Conversely, suppose $f \in T$ and satisfies (2.1). For $|z| = r$, $0 \leq r < 1$, we have (2.4) by (2.1), since $r^{m-1} < 1$. So we have

$$\left| \sum_{m=2}^{\infty} \frac{(m+n-1)![n+1+\lambda(m-1)]}{(n+1)!(m-1)!} a_m z^{m-1} \right|$$

$$\leq \sum_{m=2}^{\infty} \frac{(m+n-1)![n+1+\lambda(m-1)]}{(n+1)!(m-1)!} a_m r^{m-1}$$

$$< (B-A) - B \sum_{m=2}^{\infty} \frac{(m+n-1)![n+1+\lambda(m-1)]}{(n+1)!(m-1)!} a_m r^{m-1}$$

$$\leq \left|(B-A) - B\sum_{m=2}^{\infty} \frac{(m+n-1)![n+1+\lambda(m-1)]}{(n+1)!(m-1)!} a_m z^{m-1}\right|$$

which gives (2.2) and hence follows that

$$(1-\lambda)\frac{D^n f(z)}{z} + \lambda\frac{D^{n+1} f(z)}{z} = \frac{1+Aw(z)}{1+Bw(z)},$$

$$w \in H,\ z \in E,\ -1 \leq A < B \leq 1.$$

That is, $f \in S(n, \lambda, A, B)$.

Corollary 1. If $f \in T$ is in $S(n, \lambda, A, B)$, then

$$a_m \leq \frac{(n+1)!(m-1)!(B-A)}{(m+n-1)![n+1+\lambda(m-1)](1+B)}$$

for $m \geq 2$. The equality holds for the functions f given by

$$f(z) = z - \frac{(n+1)!(m-1)!(B-A)}{(m+n-1)![n+1+\lambda(m-1)](1+B)} z^m, \quad z \in E.$$

3. Distortion and Covering Theorems.

<u>Theorem 2</u>. Let the function $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geq 0$ be in the class $S(n, \lambda, A, B)$, then

$$r - \frac{(B-A)}{(1+B)(n+1+\lambda)} r^2 \leq |f(z)| \leq r + \frac{(B-A)}{(1+B)(n+1+\lambda)} r^2, \quad (|z| = r),$$

with equality for $f(z) = z - \frac{(B-A)}{(1+B)(n+1+\lambda)} z^2$, $(z = \pm r)$.

<u>Proof</u>. Since $(m+n-1)!/(m-1)!$ is an increasing function of m, therefore it is from (2.1) we have

$$\sum_{m=2}^{\infty} a_m \leq \frac{(B-A)}{(1+B)(n+1+\lambda)} . \tag{3.1}$$

Thus

$$|f(z)| \leq |z| + \sum_{m=2}^{\infty} a_m |z|^m$$

$$\leq r + r^2 \sum_{m=2}^{\infty} a_m$$

$$\leq r + \frac{(B-A)}{(1+B)(n+1+\lambda)} r^2, \quad \text{for} \quad |z| = r < 1.$$

Similarly

$$|f(z)| \geq |z| - \sum_{m=2}^{\infty} a_m |z|^m$$

$$\geq r - r^2 \sum_{m=2}^{\infty} a_m$$

$$\geq r - \frac{(B-A)}{(1+B)(n+1+\lambda)} r^2, \quad \text{for} \quad |z| = r < 1.$$

Theorem 3. Let the function $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geq 0$ be in the class $S(n, \lambda, A, B)$ and $\lambda \geq n+1$, then

$$1 - \frac{2(B-A)}{(1+B)(n+1+\lambda)} r \leq |f'(z)| \leq 1 + \frac{2(B-A)}{(1+B)(n+1+\lambda)} r \quad (|z| = r).$$

Equality holds for $f(z) = z - \frac{(B-A)}{(1+B)(n+1+\lambda)} z^2 \quad (z = \pm r)$.

Proof. As the proof of Theorem 2, in view of the inequality (2.1), we have

$$\sum_{m=2}^{\infty} (n+1-\lambda)a_m + \sum_{m=2}^{\infty} \lambda m a_m \leq \frac{(B-A)}{(1+B)}.$$

Hence

$$(3.2) \qquad \sum_{m=2}^{\infty} m a_m \leq \frac{1}{\lambda}\left[\frac{(B-A)}{1+B} - (n+1-\lambda)\sum_{m=2}^{\infty} a_m\right]$$

$$\leq \frac{1}{\lambda}\,\frac{(B-A)}{1+B}\left[1 - \frac{n+1-\lambda}{n+1+\lambda}\right]$$

$$= \frac{2(B-A)}{(1+B)(n+1+\lambda)},$$

by (3.1). Which implies that

$$|f'(z)| \leq 1 + \sum_{m=2}^{\infty} m a_m |z|^{m-1} \leq 1 + r\sum_{m=2}^{\infty} m a_m$$

$$\leq 1 + \frac{2(B-A)}{(1+B)(n+1+\lambda)}r.$$

On the other hand

$$|f'(z)| \geq 1 - \sum_{m=2}^{\infty} m a_m \leq 1 - \frac{2(B-A)}{(1+B)(n+1+\lambda)} r.$$

For the extreme points of $S(n, \lambda, A, B)$ we have

<u>Theorem 4</u>. Set $f_1(z) = z$ and

$$f_m(z) = z - \frac{(B-A)(n+1)!(m-1)!}{(1+B)(m+n-1)![n+1+\lambda(m-1)]} z^m,$$

$(m = 2, 3, 4, \ldots)$. Then $f \in S(n, \lambda, A, B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z), \text{ where } \mu_m \geq 0 \text{ and } \sum_{m=1}^{\infty} \mu_m = 1.$$

<u>Proof</u>. Suppose $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$. Then

$$\sum_{m=2}^{\infty} \frac{(m+n-1)![n+1+\lambda(m-1)]}{(n+1)!(m-1)!} \cdot \frac{(B-A)(n+1)!(m-1)!}{(1+B)(m+n-1)![n+1+\lambda(m-1)]} \mu_m$$

$$= \frac{B-A}{1+B} \sum_{m=2}^{\infty} \mu_m = (1-\mu_1)\frac{B-A}{1+B} \leq \frac{B-A}{1+B},$$

and hence $f \in S(n, \lambda, A, B)$ by Theorem 1. Conversely, let $f(z) = z - \sum_{m=2}^{\infty} a_m z^m \in S(n, \lambda, A, B)$. Then

$$a_m \leq \frac{(n+1)!(m-1)!(B-A)}{(m+n-1)![n+1+\lambda(m-1)](1+B)}, \quad m = 2, 3, 4, \ldots$$

by Corollary 1. Set

$$\mu_m = \frac{(m+n-1)![n+1+\lambda(m-1)](1+B)}{(n+1)!(m-1)!(B-A)} a_m, \quad m = 2, 3, 4, \ldots,$$

and define $\mu_1 = 1 - \sum_{m=2}^{\infty} \mu_m$. From Theorem 1, we have $\sum_{m=2}^{\infty} \mu_m \leq 1$ and so $\mu_1 \geq 0$. Since $\mu_m f_m(z) = \mu_m z - a_m z^m$, $\sum_{m=1}^{\infty} \mu_m f_m(z) = z - \sum_{m=2}^{\infty} a_m z^m = f(z)$.

4. Radii of Univalence, Starlikeness and Convexity.

Theorem 5. If $f \in S(n, \lambda, A, B)$, then $\mathrm{Re}\{\frac{D^{n+1}f(z)}{z}\} > 0$ for $|z| < r(n, \lambda, A, B)$, where

$$r(n, \lambda, A, B) = \inf_m \{\frac{[n+1+\lambda(m-1)](1+B)}{(m+n)(B-A)}\}^{1/m-1}.$$

Proof. It is sufficient to show that $|\frac{D^{n+1}f(z)}{z} - 1| < 1$ for $|z| < r(n, \lambda, A, B)$. We have

$$|\frac{D^{n+1}f(z)}{z} - 1| \le \sum_{m=2}^{\infty} \frac{(m+n)!}{(n+1)!(m-1)!} a_m |z|^{m-1} .$$

Hence $|\frac{D^{n+1}f(z)}{z} - 1| < 1$ if

$$\sum_{m=2}^{\infty} \frac{(m+n)!}{(n+1)!(m-1)!} a_m |z|^{m-1} < 1. \tag{4.1}$$

From Theorem 1, it is easily to see that (4.1) is true if

$$\frac{(m+n)!}{(n+1)!(m-1)!} a_m |z|^{m-1} \le \frac{(m+n-1)![n+1+\lambda(m-1)](1+B)}{(n+1)!(m-1)!(B-A)} a_m ,$$

that is

(4.2) $$|z| \leq \{\frac{[n+1+\lambda(m-1)](1+B)}{(m+n)(B-A)}\}^{1/m-1}, \quad m = 2, 3, 4, \ldots .$$

Writting $|z| = r(n, \lambda, A, B)$ in (4.2) the result follows.

Similar, we have

Theorem 6. If $f \in s(n, \lambda, A, B)$, then $\operatorname{Re} f'(z) > 0$ for $|z| < r(n, \lambda, A, B)$, where

$$r(n, \lambda, A, B) = \inf_m \{\frac{(m+n-1)![n+1+\lambda(m-1)](1+B)}{(n+1)!m!(B-A)}\}^{1/m-1},$$

$$m = 2, 3, 4, \ldots .$$

The estimate is sharp for the function

$$f(z) = z - \frac{(B-A)(n+1)!(m-1)!}{(1+B)(m+n-1)![n+1+\lambda(m-1)]} z^m ;$$

for some m.

Theorem 7. If $f \in S(n, \lambda, A, B)$, then $\operatorname{Re}\{zf'(z)/f(z)\} > 0$ for $|z| < r(n, \lambda, a, B)$, where $r(n, \lambda, A, B)$ is as in

Theorem 6.

Theorem 8. If $f \in S(n, \lambda, A, B)$, then $\mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} > 0$ for $|z| < r(n, \lambda, A, B)$, where

$$r(n, \lambda, A, B) = \inf_m \{\frac{(m+n-1)![n+1+\lambda(m-1)](1+B)}{(n+1)!m!(B-A)m}\}^{1/m-1},$$

$$m = 2, 3, 4, \ldots .$$

The estimates in the above two theorems are sharp for the function

$$f(z) = z - \frac{(B-A)(n+1)!(m-1)!}{(1+B)(m+n-1)![n+1+\lambda(m-1)]} z^m,$$

for some m.

Theorem 9. If $f \in S(n, \lambda, A, B)$, then f satisfies

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1 + Cw(z)}{1 + Dw(z)}, \quad -1 \leq C < D \leq 1, \tag{4.3}$$

$n \in N \cup \{0\}$ and $w \in H$, for $|z| < r(n, \lambda, A, B, C, D)$, where

$$r(n, \lambda, A, B, C, D) = \inf_{m}\left\{\frac{[n+1+\lambda(m-1)](B-A)(D-C)}{c_m(1+B)}\right\}^{1/m-1},$$

$$m = 2, 3, 4, \ldots, \quad \text{and} \quad c_m = (D+1)(n+m) - (C+1)(n+1).$$

Proof. From the proof of Theorem 1 in [3] we know that f satisfies (4.3) if

$$\sum_{m=2}^{\infty} \frac{(m+n-1)!}{(n+1)!(m-1)!} c_m a_m r^{m-1} < D - C . \tag{4.4}$$

By inequality (2.1), it is easily to see that (4.4) holds if $c_m r^{m-1} \leq \frac{[n+1+\lambda(m-1)](B-A)(D-C)}{1+B}$, and this completes the proof of Theorem 9. When $n = 0$, $c = -1$, $A = 2\alpha-1$, $B = D = 1$, Theorem 9 reduces to Theorem 6 of [2].

Similar, we can generalize [2, Theorem 7] as following:

Theorem 10. If $f \in S(n, \lambda, A, B)$, then f satisfies

$$\frac{D^{n+1}(zf'(z))}{D^n(zf'(z))} = \frac{1 + Cw(z)}{1 + Dw(z)}, \quad -1 \leq C < D \leq 1,$$

$n \in N \cup \{0\}$ and $w \in H$, for

$$|z| \leq \inf_m \{\frac{[n+1+\lambda(m-1)](B-A)(D-C)}{mc_m(1+B)}\}^{1/m-1}, \quad m = 2, 3, 4, \ldots .$$

5. Modified Hadamard Product.

Theorem 11. If $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geq 0$, $g(z) = z - \sum_{m=2}^{\infty} b_m z^m$, $b_m \geq 0$ are elements of $S(n, \lambda, A, B)$ and $S(n, \lambda, C, D)$ respectively. Then the modified Hadamard product $h(z) = f(z)*g(z) = z - \sum_{m=2}^{\infty} a_m b_m z^m$ is an element of $S(n, \lambda, 1 - \frac{2(B-A)(D-C)}{(n+1+\lambda)(1+B)(1+D)}, 1)$.

Proof. From Theorem 1, we have

$$(5.1) \qquad \sum_{m=2}^{\infty} \frac{(m+n-1)![n+1+\lambda(m-1)](1+B)}{(n+1)!(m-1)!(B-A)} a_m \leq 1$$

and

(5.2) $$\sum_{m=2}^{\infty} \frac{(m+n-1)![n+1+\lambda(m-1)](1+D)}{(n+1)!(m-1)!(D-C)} b_m \leq 1$$

we want to find $\beta = \beta(n, \lambda, A, B)$ such that

(5.3) $$\sum_{m=2}^{\infty} \frac{2(m+n-1)![n+1+\lambda(m-1)]}{(n+1)!(m-1)!(1-\beta)} a_m b_m \leq 1 .$$

From (5.1) and (5.2) by means of Cauchy-Schwarz inequality we obtain

(5.4) $$\sum_{m=2}^{\infty} \frac{(m+n-1)![n+1+\lambda(m-1)]\sqrt{(1+B)(1+D)}}{(n+1)!(m-1)!\sqrt{(B-A)(D-C)}} \sqrt{a_m b_m} \leq 1 .$$

Hence (5.3) will be satisfied if

$$\sqrt{a_m b_m} \leq \frac{(1-\beta)\sqrt{(1+B)(1+D)}}{2\sqrt{(B-A)(D-C)}} .$$

From (5.4) it follows that

$$\sqrt{a_m b_m} \leq \frac{(n+1)!(m-1)!\sqrt{(B-A)(D-C)}}{(m+n-1)![n+1+\lambda(m-1)]\sqrt{(1+B)(1+D)}} \quad \text{for each } m.$$

Therefore (5.3) will be satisfied if

$$(5.5) \qquad \frac{(n+1)!(m-1)!}{(m+n-1)![n+1+\lambda(m-1)]} \leq \frac{(1-\beta)(1+B)(1+D)}{2(B-A)(D-C)}$$

for all m. That is

$$(5.6) \qquad \beta \leq 1 - \frac{2(n+1)!(m-1)!(B-A)(D-C)}{(m+n-1)![n+1+\lambda(m-1)](1+B)(1+D)}.$$

The right hand side of (5.6) is an increasing function of m for $m = 2, 3, 4, \ldots$. Therefore, setting $m = 2$ in (5.6) we get

$$\beta \leq 1 - \frac{2(B-A)(D-C)}{(n+1+\lambda)(1+B)(1+D)} .$$

The result is sharp, with equality when

$$f(z) = z - \frac{(B-A)}{(n+1+\lambda)(1+B)} z^2,$$

and

$$g(z) = z - \frac{(D-C)}{(n+1+\lambda)(1+D)} z^2 .$$

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