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NOTES ON CERTAIN CLASSES OF ANALYTIC FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVES

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1. INTRODUCTION

Let A_p denote the class of functions of the form

(1.1) $$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in N = \{1,2,3, \ldots\})$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$. For functions

(1.2) $$f_j(z) = z^p + \sum_{n=p+1}^{\infty} a_{n,j} z^n \quad (j = 1,2)$$

belonging to A, we denote by $f_1 * f_2(z)$ the convolution of $f_1(z)$ and $f_2(z)$, that is,

(1.3) $$f_1 * f_2(z) = z^p + \sum_{n=p+1}^{\infty} a_{n,1} a_{n,2} z^n.$$

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Using the convolution, we define the Ruscheweyh derivative $D^{\alpha+p-1}f(z)$ by

(1.4) $$D^{\alpha+p-1}f(z) = \frac{z^p}{(1-z)^{\alpha+p}} * f(z) \qquad (\alpha > -p)$$

for $f(z) \in A_p$. A function $f(z)$ belonging to A_p is said to be in the class $R(\alpha+p-1)$ if it satisfies

(1.5) $$\mathrm{Re}\left\{\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)}\right\} > \frac{\alpha+p-1}{\alpha+p} \quad (z \in U),$$

where $\alpha > -p$.

For an integer α greater than $-p$, Soni [4] proved

THEOREM A. $R(\alpha+p) \subset R(\alpha+p-1)$.

for $\alpha = n \in N_0 = \{0, 1, 2, \dots\}$ and $p = 1$, Singh and Singh [3] proved

THEOREM B. $R(n+1) \subset R(n)$.

In the present paper, we improve the above results.

2. PROPERTIES OF THE CLASS $R(\alpha+p-1)$

We begin with the statement of the following lemma due to Miller and

Mocanu [2].

LEMMA 1. Let $\varphi(u, v)$ be a complex valued function,

$$\varphi\colon D \to C,\ D \subset C^2 \quad (C \text{ is the complex plane}),$$

and let $u = u_1+iu_2$, $v = v_1+iv_2$. Suppose that the function $\varphi(u, v)$ satisfies

(i) $\varphi(u,v)$ is continuous in D;

(ii) $(1, 0) \in D$ and $\mathrm{Re}\{\varphi(1, 0)\} > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{1+u_2^2}{2}$, $\mathrm{Re}\{\varphi(iu_2, v_1)\} \leq 0$.

Let $q(z) = 1 + q_1 z + q_2 z^2 + \cdots$ be regular in U such that $(q(z), zq(z)) \in D$ for all $z \in U$. If

$$\mathrm{Re}\{\varphi(q(z), zq'(z))\} > 0 \quad (z \in U),$$

then $\mathrm{Re}(q(z))\} > 0 \quad (z \in U)$.

Applying the above lemma, we derive

THEOREM 1. If $f(z) \in R(\alpha+p)$, $\alpha \geq 1-p$, $p \in N$, then

$$(2.1)\qquad \mathrm{Re}\left\{\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)}\right\} > \beta(\alpha,\ p) \quad (z \in U),$$

where

$$(2.2)\qquad \beta(\alpha,\ p) = \frac{2(\alpha+p)-3+\sqrt{4(\alpha+p)^2-4(\alpha+p)+9}}{4(\alpha+p)}.$$

PROOF. We define the function $q(z)$ by

$$(2.3)\qquad \frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} = \beta + (1-\beta)q(z),$$

where $\beta = \beta(\alpha,\ p)$. Noting that

$$(2.4)\qquad z(D^{\alpha+p-1}f(z))' = (\alpha+p)D^{\alpha+p}f(z) - \alpha(D^{\alpha+p-1}f(z)) \quad (\alpha > -p),$$

we have, from (2.3), that

$$(2.5)\qquad \frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)} = \frac{1}{\alpha+p-1}\left\{1 + (\alpha+p)(\beta+(1-\beta)q(z)) + \frac{(1-\beta)zq'(z)}{\beta+(1-\beta)q(z)}\right\},$$

or

$$(2.6)\qquad \mathrm{Re}\left\{\frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)} - \frac{\alpha+p}{\alpha+p+1}\right\}$$

$$= \mathrm{Re}\frac{1}{\alpha+p+1}\left\{(1-\alpha-p) + (\alpha+p)(\beta+(1-\beta)q(z)) + \frac{(1-\beta)zq'(z)}{\beta+(1-\beta)q(z)}\right\}$$

$$> 0.$$

Letting

$$\varphi(u,v) = \frac{1}{\alpha+p+1}\left\{(1-\alpha-p) + (\alpha+p)(\beta+(1-\beta)u) + \frac{(1-\beta)v}{\beta+(1-\beta)u}\right\}, \tag{2.7}$$

we see that

(i) $\varphi(u, v)$ is continuous in $D = (C - \{\frac{\beta}{\beta-1}\}) \times C$;

(ii) $(1, 0) \in D$ and $\mathrm{Re}\{\varphi(1, 0)\} = \frac{1}{\alpha+p+1} > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{1+u_2^2}{2}$,

$$\mathrm{Re}\{\varphi(iu_2,v_1)\} = \frac{1}{\alpha+p+1}\{(1-\alpha-p)+\beta(\alpha+p) + \frac{\beta(1-\beta)v_1}{\beta^2+(1-\beta)^2u_2^2}\}$$

$$\leq \frac{1}{\alpha+p+1}\{(1-\alpha-p)+\beta(\alpha+p) - \frac{\beta(1-\beta)(1+u_2^2)}{2(\beta^2+(1-\beta)^2u_2^2)}\}.$$

$$= \frac{1}{\alpha+p+1}\{2(\alpha+p)\beta^2-[2(\alpha+p)-3]\beta-1)(\beta+(\beta-1)u_2^2)$$

$$+ (1-\beta)[2(\alpha+p)(\beta-1)+1]u_2^2\}$$

$$= \frac{(1-\beta)[2(\alpha+p)(\beta-1)+1]u_2^2}{\alpha+p+1}$$

$$\leq 0$$

because $2(\alpha+p)\beta^2 - [2(\alpha+p)-3]\beta-1 = 0$, and under the condition $\alpha+p \geq 1$, we have

$$0 < \beta \leq 1 - \frac{1}{2(\alpha+p)} < 1.$$

Therefore, the function $\varphi(u, v)$ satisfies the conditions in Lemma 1. Since $\mathrm{Re}\{q(z)\} > 0$ $(z \in U)$, we conclude that

$$(2.8) \quad \mathrm{Re}\left\{\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)}\right\} > \beta = \frac{2(\alpha+p)-3+\sqrt{4(\alpha+p)^2-4(\alpha+p)+9}}{4(\alpha+p)}$$

REMARK 1. Since

$$(2.9) \quad \mathrm{Re}\left\{\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)}\right\} > \frac{2(\alpha+p)-3+\sqrt{4(\alpha+p)^2-4(\alpha+p)+9}}{4(\alpha+p)} > \frac{\alpha+p-1}{\alpha+p}$$

for $\alpha \geq 1-p$, Theorem 1 is the improvement of Theorem A due to Soni [4].

REMARK 2. For $\alpha = n \in N_0$ and $p = 1$, we have

$$(2.10) \quad \mathrm{Re}\left\{\frac{D^{n+1}f(z)}{D^{n}f(z)}\right\} > \frac{2n-1+\sqrt{4n^2+4n+9}}{4(n+1)} > \frac{n}{n+1}.$$

Therefore, Theorem 1 is also the improvement of Theorem B by Singh and Singh [3].

Taking $\alpha = 2-p$ in Theorem 1, we have

COROLLARY 1. If $f(z) \in R(2)$, then

$$\text{(2.11)} \qquad \mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} > \frac{\sqrt{17}-3}{4} \quad (z \in U),$$

that is, $f(z)$ is p–valently convex of order $\frac{\sqrt{17}-3}{4}$.

PROOF. For $\alpha = 2-p$, we have

$$\text{(2.12)} \qquad \mathrm{Re}\{\frac{D^2f(z)}{D^1f(z)}\} = \mathrm{Re}\{1 + \frac{1}{2}\frac{zf''(z)}{f'(z)}\} > \frac{1+\sqrt{17}}{8}.$$

Therefore, (2.12) leads to

$$\text{(2.13)} \qquad \mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} > \frac{1+\sqrt{17}}{4} - 1 = \frac{\sqrt{17}-3}{4}.$$

3. AN APPLICATION OF JACK'S LEMMA

In order to prove our next result, ww need the following lemma by Jack [1].

LEMMA 2. Let $w(z)$ be regular in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in U$, then

$$z_0 w'(z_0) = kw(z_0),$$

where k is real and $k \geq 1$.

THEOREM 2. If $f(z) \in A_p$ satisfies

$$\text{(3.1)} \qquad \mathrm{Re}\left\{\frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)}\right\} > \frac{1+(\alpha+p-1)^2}{\alpha+p+1} \quad (z \in U),$$

and $\alpha + p \geq 1$ then

$$\text{(3.2)} \qquad \mathrm{Re}\left[\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)}\right]^{\frac{1}{2}} > \frac{\alpha+p-1}{\alpha+p} \quad (z \in U).$$

PROOF. For $f(z) \in A_p$ satisfying (3.1), we define the function $w(z)$ by

$$\text{(3.3)} \mathrm{Re}\left[\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)}\right]^{\frac{1}{2}} = \frac{\alpha+p-1}{\alpha+p} + \frac{1}{\alpha+p}\,\frac{1-w(z)}{1+w(z)} = \frac{(\alpha+p)+(\alpha+p-2)w(z)}{(\alpha+p)(1+w(z))}.$$

Then $w(z)$ is regular in U with $w(0) = 0$ and $w(z) \neq -1$ $(z \in U)$.

After taking the logarithmic derivative of both sides of (3.3), using (2.4) and (3.3) we obtain

$$\text{(3.4)} \qquad \frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)}$$

$$= \frac{1}{\alpha+p+1}\left\{1+\left(\frac{(\alpha+p)+(\alpha+p-2)w(z)}{1+w(z)}\right)^2 - \frac{4zw'(z)}{(1+w(z))((\alpha+p)+(\alpha+p-2)w(z))}\right\}.$$

If there exists a point $z_0 \in U$ such that

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1,$$

then by Lemma 2, we have

$$z_0 w'(z_0) = kw(z_0) \qquad (k \geq 1).$$

It follows from

$$\text{Re}\left\{\frac{1}{1+w(z_0)}\right\} = \frac{1}{2}$$

and

$$\text{Re}\left\{\frac{w(z_0)}{1+w(z_0)}\right\} = \frac{1}{2}$$

that

(3.5) $$\left\{\frac{D^{\alpha+p+1}f(z_0)}{D^{\alpha+p}f(z_0)}\right\}$$

$$= \mathrm{Re}\frac{1}{\alpha+p+1}\left\{1+\left(\frac{(\alpha+p)+(\alpha+p-2)w(z_0)}{1+w(z_0)}\right)^2 - \frac{4kw'(z_0)}{(1+w(z_0))((\alpha+p)+(\alpha+p-2)w(z_0))}\right\}.$$

Let $w(z_0) = e^{i\theta}$, we have

$$(1 + w(z_0))^2 = (1 + e^{i\theta})^2 = (e^{-\frac{i\theta}{2}} + e^{\frac{i\theta}{2}})^2 e^{i\theta}$$

$$= (2\cos\frac{\theta}{2})^2 e^{i\theta} = 2(1 + \cos\theta)w(z_0).$$

It follows that

(3.6) $$\mathrm{Re}\left[\frac{(\alpha+p)+(\alpha+p-2)w(z_0)}{1+w(z_0)}\right]^2$$

$$= \mathrm{Re}\left\{\frac{(\alpha+p)^2}{2(1+\cos\theta)e^{i\theta}}\right\} + \mathrm{Re}\left\{\frac{2(\alpha+p)(\alpha+p-2)}{2(1+\cos\theta)}\right\} + \mathrm{Re}\left\{\frac{(\alpha+p-2)^2 e^{i\theta}}{2(1+\cos\theta)}\right\}$$

$$= \frac{\{(\alpha+p)^2+(\alpha+p-2)^2\}\cos\theta+2(\alpha+p)(\alpha+p-2)}{2(1+\cos\theta)}$$

$$\leq \frac{(\alpha+p)^2+(\alpha+p-2)^2+2(\alpha+p)(\alpha+p-2)}{4} = (\alpha+p-1)^2,$$

since $(\alpha+p)^2 + (\alpha+p-2)^2 \geq 2(\alpha+p)(\alpha+p-2)$ and $\frac{Ax+B}{1+x}$ is an increasing function of x if $A \geq B$.

Next we have

$$(3.7)\quad \mathrm{Re}\left\{\frac{w(z_0)}{(1+w(z_0))((\alpha+p)+(\alpha+p-2)w(z_0))}\right\}$$

$$= \frac{1}{|(\alpha+p)+(\alpha+p-2)w(z_0)|^2}\,\mathrm{Re}\left\{\frac{w(z_0)[(\alpha+p)+(\alpha+p-2)\overline{w(z_0)}]}{1+w(z_0)}\right\}$$

$$= \frac{1}{|(\alpha+p)+(\alpha+p-2)w(z_0)|^2}\left\{(\alpha+p)\frac{1}{2} + (\alpha+p-2)\frac{1}{2}\right\} \geq 0.$$

From (3.5), (3.6) and (3.7) we obtain

$$\mathrm{Re}\left\{\frac{D^{\alpha+p+1}f(z_0)}{D^{\alpha+p}f(z_0)}\right\} \leq \frac{1+(\alpha+p-2)^2}{\alpha+p+1}.$$

This contradicts our condition (3.1). Therefore, we conclude that $w(z)$ satisfies $|w(z)| < 1$ for all $z \in U$, which completes the proof of our theorem.

Letting $\alpha = 2-p$, Theorem 2 gives

COROLLARY. If $f(z) \in A_p$ satisfies

$$(3.8)\qquad \mathrm{Re}\left\{\frac{D^3f(z)}{D^2f(z)}\right\} > \frac{2}{3} \quad (z \in U),$$

then

$$\mathrm{Re}\sqrt{2 + \frac{zf''(z)}{f'(z)}} > \frac{\sqrt{2}}{2} \qquad (z \in U). \tag{3.9}$$

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