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On the Pullback of a Differential Operator and its Application

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1. Introduction

In [4], Garrett showed nice decomposition of a pull back of Siegel's Eisenstein series. We generalize his result to the space of vector valued modular forms of weight $\det^k\otimes\mathrm{Sym}^l$. In this case, there is no Siegel's Eisenstein series in the sense that constant term of vector valued modular forms vanish. (Cf. Weissauer[8, Satz 1].) To avoid this difficulty, we construct, in the section 2, a differential operator whose pullback sends modular forms to smaller degree ones. Next, we construct Poincaré series of vector valued modular forms of weight $\det^k\otimes\mathrm{Sym}^l$. These result together with coset decomposition by Garrett [4, Sect. 2-3] yields a desired pullback formula.

Notation

We put $\Gamma_n = Sp(n,\mathbf{Z})$. Let ρ be a representation of $GL(n,\mathbf{C})$ with a representation space W. Let H_n be the Siegel upper half plane of degree n. The W-valued C^∞-modular form f of degree n and of weight ρ is a C^∞-function from H_n to W satisfying

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Because of the first author's misestimation of time to write this paper, the second and the third authors did not check the manuscript. All of the inaccuracies and other faults are due to the first author although, needless to say, the effective results are consequence of cooperation among three.

$$(f|_{\rho}M)(Z) = f(Z)$$

for all $Z \in H_n$ and $M \in \Gamma_n$ where

$$(f|_{\rho}\begin{pmatrix} A & B \\ C & D \end{pmatrix})(Z) = \rho((CZ+D)^{-1})(f(Z))$$

for $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathrm{R})$. The space of all such functions is denoted by $M_{\rho}^{\infty}(W)$. When ρ is a representation $\det^k \otimes \mathrm{Sym}^l$ of $GL(n,\mathrm{C})$, we write $|_{\rho}$ and $M_{\rho}^{\infty}(W)$ as $|_{k,l,n}$ and $M_{k,l,n}^{\infty}(W)$, respectively. We note $M_{k,l,n}^{\infty}(W) = \{0\}$ unless $nk \equiv l \bmod 2$. We put

$$M_{k,l,n}(W) = \{ f \in M_{k,l,n}^{\infty}(W) \mid f \text{ is holomorphic on } H_n \text{ (and its cusp)}\}$$

and

$$S_{k,l,n}(W) = \{ f \in M_{k,l,n}(W) \mid f \text{ is a cuspform. } \}$$

We omit the subscript '$,n$' when there is no fear of confusion. For a vector space W, we denote by $W^{(l)}$ its l-th symmetric tensor product. We identify $W^{(0)}$ with C. Let $x = (x_1,\ldots,x_n)$ be a row vector consisting of n indeterminates. Through out this paper, we put $V = \mathrm{C}x_1 \oplus \ldots \oplus \mathrm{C}x_n$. We identify $V^{(l)}$ with $\mathrm{C}[x_1,\ldots,x_n]_{(l)}$ where the subscript (l) stands for homogeneous polynomials of degree l. Then $GL(n,\mathrm{C})$ acts on $V^{(l)}$ by

$$(gv)(x) = \det g^k v(xg)$$

for $g \in GL(n,\mathrm{C})$ and $v \in V^{(l)}$. This is isomorphic to $\det^k \otimes \mathrm{Sym}^l$ and we always use this realization. We also identify $C^{\infty}(H_n, V^{(l)})$ with $C^{\infty}(H_n)[x_1,\ldots,x_n]_{(l)}$.

2. Differential Operator

Let $Z = (z_{ij})$ be a variable on H_n. For an integer $l \geq 1$ and an $f \in C^\infty(H_n, V^{(l)})$, we put

$$Df = \left(\frac{1}{2\pi i}\frac{\partial}{\partial Z}f\right)[x],$$

$$Nf = \left(-\frac{1}{4\pi}(\operatorname{Im} Z)^{-1}f\right)[x].$$

and

$$\delta_k f = kNf + Df. \tag{2.1}$$

Here, as usual, $A[x] = xA\,{}^t x$ and $\frac{\partial}{\partial Z} = \left(\frac{1+\delta_{ij}}{2}\frac{\partial}{\partial z_{ij}}\right)_{1 \leq i,j \leq n}$. Then, Df, Nf and $\delta_k f$ are $V^{(l+2)}$-valued functions. For an integer $l \geq 0$, we put

$$k^{[l]} = \begin{cases} k(k+1)\ldots(k+l-1) & (l>0) \\ 1 & (l=0). \end{cases}$$

Note $A^{[n]} = (-1)^n(-A-n+1)^{[n]}$. We also have

$$(A+B)^{[n]} = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} A^{[r]} B^{[n-r]}$$

and

$$\sum_{r=0}^{n} \frac{n!}{r!(n-r)!}(A-2r)(-A)^{[r]}(A-2n)^{[n-r]} = \begin{cases} A & (n=0) \\ 0 & (n \neq 0). \end{cases} \tag{2.2}$$

Lemma 2.1. The operator δ_{k+l} maps $M^\infty_{k,l,n}(V^{(l)})$ to $M^\infty_{k,l+2,n}(V^{(l+2)})$. For each integer $l \geq 0$, we have

$$\delta^r_{k+l} = \sum_{i=0}^{r} (k+l+r-i)^{[i]} \binom{r}{i} N^i D^{r-i}. \tag{2.3}$$

Proof. The former part follows from [9] and

$$(\mathrm{Im}\, M\langle Z\rangle)^{-1} = (\mathrm{Im}\, Z)^{-1}[cZ+d]-2i(cZ+d)\,{}^t c,$$

where $M=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_n$. Note

$$D((\mathrm{Im}\, Z)^{-1}[x]) = \frac{1}{4\pi}((\mathrm{Im}\, Z)^{-1}[x])^2.$$

Since D is a derivation and N is essentially a multiplication,

$$DN^l = -lN^{l+1}+N^l D. \tag{2.4}$$

Using induction on r, we have (2.3). □

It is remarkable that the differential operator acting on $M^{\infty}_{k,l,n}(V^{(l)})$ depends only on $k+l$. We note (2.1) and (2.3) do not explicitly contain n. Let $G_j(t)$ be a formal power series of t defined by

$$G_j(t) = \sum_{l=0}^{\infty}\frac{t^l}{l!\,j^{[l]}}\delta_j^l.$$

Following Cohen[2, Sect. 7], we have

$$G_j(t) = e^{tN}\sum_{l=0}^{\infty}\frac{t^l}{l!\,j^{[l]}}D^l$$

In what follows, we put $n = p+q$ where p and q are positive integers. Let $V_1 = \mathbf{C}x_1\oplus\ldots\oplus\mathbf{C}x_p$ and $V_2 = \mathbf{C}x_{p+1}\oplus\ldots\oplus\mathbf{C}x_n$ be two subspaces of V. We note $V_1^{(l)}$ and $V_2^{(l)}$ are subspaces of $V^{(l)}$ which are stable under the $GL(p)\times GL(q)$. Let X be any map $X: A \to C^{\infty}(H_n, V^{(l)})$ for any set A. We define two maps $X_{\uparrow}: A \to C^{\infty}(H_n, V_1^{(l)})$ and $X_{\downarrow}: A \to C^{\infty}(H_n, V_2^{(l)})$ by

$$(X_\uparrow(a))(x_1,\ldots,x_p) = (X(a))(x_1,\ldots,x_p,0,\ldots,0)$$

and

$$(X_\downarrow(a))(x_{p+1},\ldots,x_n) = (X(a))(0,\ldots 0,x_{p+1},\ldots,x_n).$$

Let d^* be the pullback of diagonal embedding $d{:}H_p\times H_q \to H_n$. Now, for each $l\geq 0$, define an operator

$$L^{(l)}{:}\mathrm{Hol}(H_n,\mathrm{C}) \to \mathrm{Hol}(H_p\times H_q, V^{(2l)})$$

inductively by

$$d^*\sum_{l=0}^{\infty}\frac{t^l}{l!\,k^{[l]}}D^l = \sum_{l=0}^{\infty}\left(\sum_{\lambda=0}^{\infty}\frac{t^\lambda}{\lambda!(k+l)^{[\lambda]}}D_\uparrow^\lambda\right)\left(\sum_{\lambda=0}^{\infty}\frac{t^\lambda}{\lambda!(k+l)^{[\lambda]}}D_\downarrow^\lambda\right)t^lL^{(l)}. \tag{2.5}$$

Lemma 2.2.

$$L^{(l)} = \frac{1}{k^{[k]}}d^*\sum_{0\leq 2\nu\leq l}\frac{1}{\nu!(l-2\nu)!(2-j-l)^{[\nu]}}(D_\uparrow D_\downarrow)^\nu(D-D_\uparrow-D_\downarrow)^{l-2\nu} \tag{2.6}$$

Proof. Since $D_\uparrow$ and $D_\downarrow$ commute,

$$\left(\sum_{l=0}^{\infty}\frac{1}{l!\,k^{[l]}}D_\uparrow^lt^l\right)\left(\sum_{l=0}^{\infty}\frac{1}{l!\,k^{[l]}}D_\downarrow^lt^l\right)$$
$$= \sum_{r=0}^{\infty}\frac{1}{k^{[r]}}\left(\sum_{1\leq\mu\leq r/2}\frac{1}{\mu!(r-2\mu)!\,k^{[\mu]}}(D_\uparrow+D_\downarrow)^{r-2\mu}(D_\uparrow D_\downarrow)^\mu\right)t^r.$$

Since (2.5) uniquely determines $L^{(l)}$, we have only to verify that (2.6) satisfies (2.5). The coefficient of t^l of right side is

$$\sum_{0\le j\le l/2}\ \sum_{0\le\rho\le l-2j}\frac{1}{\rho!(l-2j-\rho)!}$$

$$\times\sum_{0\le\mu\le j}\frac{1}{\mu!(j-\mu)!(k+l-\rho-2\mu)^{[\mu]}(-k-l+\rho+2\mu+2)^{[j-\mu]}}(D_\uparrow+D_\downarrow)^{\rho}(D_\uparrow D_\downarrow)^{j}(D-D_\uparrow-D_\downarrow)^{l-2j-\rho}.$$

Using (2.2), we see this is $\frac{1}{l!}D^l$. □

Note the direct sum decomposition

$$V^{(l)} = (V_1\oplus V_2)^{(l)} = \bigoplus_{a=0}^{l} V_1^{(a)}\cdot V_2^{(l-a)}$$

where $\cdot$ is a symmetric tensor product. We denote by π_a^l the projection $V^{(l)}\to V_1^{(a)}\cdot V_2^{(l-a)}$. For $f\in\mathrm{Map}(H_n,V^{(l)})$, define $\pi_a^l f$ by $(\pi_a^l f)(Z) = \pi_a^l(f(Z))$.

Lemma 2.3.

$$\pi_{a+2}^{l+2}\delta_{(1),k} = \delta_{(1),k}\pi_a^l$$

$$\pi_a^{l+2}\delta_{(2),k} = \delta_{(2),k}\pi_a^l$$

Lemma 2.4. Let $f\in M_{k,l,n}^{\infty}(V^{(l)})$. Then

$$d^*\pi_a^l f \in M_{k,a,p}^{\infty}(V_1^{(a)})\otimes M_{k,l-a,q}^{\infty}(V_2^{(l-a)}).$$

These are standard.

Proposition 2.5. Let $f\in M_{k,0,n}(\mathrm{C})$. Then

$$L^{(l)}f \in M_{k,l,p}(V_1^{(l)})\otimes M_{k,l,q}(V_2^{(l)}). \tag{2.8}$$

Proof. We use induction on l. For $l=0$, this proposition certainly holds because $L^{(0)} = d^*$. Let $l>0$. Multiplying $d^* e^{tN} = e^{t(N_\uparrow+N_\downarrow)}$ on both sides of (2.5), we have

$$d^*\sum_{l=0}^{\infty}\frac{t^l}{l!\,k^{[l]}}\delta_k^l = \sum_{l=0}^{\infty}\left(\sum_{\lambda=0}^{\infty}\frac{t^\lambda}{\lambda!(k+l)^{[\lambda]}}\delta_{k+l\uparrow}^{\lambda}\right)\left(\sum_{\lambda=0}^{\infty}\frac{t^\lambda}{\lambda!(k+l)^{[\lambda]}}\delta_{k+l\downarrow}^{\lambda}\right)t^l L^{(l)}$$

Hence there are constants $c_{l,j,a}\in\mathrm{C}$ such that

$$d^*\delta_k^l f = L^{(l)}f+\sum_{j=1}^{l}\left(\sum_{a=0}^{j}c_{j,l,a}\delta_{k+2(l-j)\uparrow}^{a}\delta_{k+2(l-j)\downarrow}^{j-a}\right)L^{(l-j)}f. \tag{2.9}$$

By Lemma 2, $(L^{(l)}f)(Z)\in V_1^{(l)}\cdot V_2^{(l)}$. Hence

$$\pi_a^{2l}L^{(l)}f = \begin{cases} L^{(l)}f & (a=l) \\ 0 & (a\neq l). \end{cases} \tag{2.10}$$

We apply π_l^{2l} on (2.9).

$$\begin{aligned} d^*\pi_l^{2l}\delta_k^l f &= L^{(l)}f+\sum_{j=1}^{l}\sum_{a=0}^{j}c_{l,j,a}\pi_l^{2l}\delta_{k+2(l-j)\uparrow}^{a}\delta_{k+2(l-j)\downarrow}^{j-a}L^{(l-j)}f \\ &= L^{(l)}f+\sum_{j=1}^{l}\sum_{a=0}^{j}c_{l,j,a}\delta_{k+2(l-j)\uparrow}^{a}\delta_{k+2(l-j)\downarrow}^{j-a}\pi_{l-2a}^{2l-2j}L^{(l-j)}f \\ &= L^{(l)}f+\sum_{1\le j\le l/2}c_{l,2j,j}\delta_{k+2l-4j\uparrow}^{j}\delta_{k+2l-4j\downarrow}^{j}L^{(l-2j)}f \end{aligned}$$

by (2.10). Hence

$$L^{(l)}f \in M_{k,l,p}^{\infty}(V_1^{(l)})\otimes M_{k,l,q}^{\infty}(V_2^{(l)})$$

by induction hypotheses and Lemma 4. By definition, $L^{(l)}f$ is a holomorphic function on $H_p\times H_q$. □

Remark 2.6. When $p=q=1$, (2.6) together with (2.8) gives a new proof of linear relation of fourier coefficients of Siegel modular forms of degree two, which is firstly proved by Maass[7]. Our proof does not use the theory of Jaccobi forms. (Cf. Eichler and Zagier[3].) Adding a certain term to D, we obtain differential operator acting on the space of Jaccobi forms. For this operator, (2.4) remains to be valid. Hence Proposition 2.5 still holds with this operator. More specifically, let $(\zeta_1,\dots,\zeta_n)$ be a variable on $\mathbf{C}^n$ and put

$$\frac{\partial}{\partial\zeta} = \left(\frac{\partial}{\partial\zeta_1},\dots,\frac{\partial}{\partial\zeta_n}\right).$$

To obtain the differential operator acting on Jaccobi forms of index m, we replace D by

$$Df = \left(\frac{1}{2\pi i}\frac{\partial}{\partial Z}f - \frac{1}{4m}\left(\frac{1}{2\pi i}\right)^{2t}\left(\frac{\partial}{\partial\zeta}\right)\left(\frac{\partial}{\partial\zeta}\right)f\right)[x].$$

3. The Kernel Function.

This section is devoted to obtain explicit form of the vector valued Poincaré series. For a symmetric positive definite matrix S, we denote by $\sqrt{S}$ the unique symmetric positive definite matrix satisfying $S = \sqrt{S}^2$. As is in the previous section, let $V = \mathbf{C}x_1\oplus\dots\oplus\mathbf{C}x_n$. Let $y=(y_1,\dots,y_n)$ be an another row vector consisting of indeterminates and put $U = \mathbf{C}y_1\oplus\dots\oplus\mathbf{C}y_n$. The inner product

$$\left(\sum_{i=1}^{n} a_i x_i, \sum_{i=1}^{n} b_i x_i\right) = \sum_{i=1}^{n} a_i\overline{b_i}$$

induces a inner product of $V^{(l)}$ defined by

$$(\alpha_1\cdots\alpha_l, \beta_1\cdots\beta_l) = \frac{1}{l!}\sum_{\tau}\prod_{j=1}^{l}(\alpha_{\tau(j)},\beta_j)$$

where α_j, $\beta_j \in V^{(l)}$ and τ runs over the symmetric group of degree l. It is also denoted by $(\ ,\ )$. This is invariant under the action of unitary matrices by Sym^l. We extend this inner product $V^{(l)}\times V^{(l)}\to \mathrm{C}$ to the map $V^{(l)}\cdot U^{(l)}\times V^{(l)}\to U^{(l)}$ complex linearly by

$$(v_1 u, v_2) = (v_1, v_2)u$$

for a monomial u of $y_1,\ldots,y_n$. If $\alpha\in V^{(l)}$ and $\beta\in V^{(l)}\cdot U^{(l)}$, we understand (α,β) to be $\overline{(\beta,\alpha)}$. We fix an isomorphism σ from V to U defined by $\sigma(x_i) = y_i$, which induces an isomorphism (also denoted by σ) from $V^{(l)}$ to $U^{(l)}$. Note

$$(v,(x\,{}^t y)^l) = \sigma(v)$$

for any $v\in V^{(l)}$. Put $\rho_{k,l} = \det^k\otimes\mathrm{Sym}^l$. We define the Petersson inner product of f, $g\in M^{\infty}_{k,l,n}(V^{(l)})$ by

$$(f,g)_{k,l} = \int_{\Gamma_n\backslash H_n}(\rho_{k,l}(\sqrt{\operatorname{Im} Z})f(Z),\rho_{k,l}(\sqrt{\operatorname{Im} Z})g(Z))\det(\operatorname{Im} Z)^{-n-1}dZ$$

whenever this integral converges. We again extend it to the map

$$(\ ,\ )_{k,l}: M^{\infty}_{k,l,n}(V^{(l)})\times M^{\infty}_{k,l,n}(V^{(l)})\cdot C^{\infty}(H_n,U^{(l)}) \to C^{\infty}(H_n,U^{(l)}).$$

Define Poincaré series by

$$P_{k,l,n}(Z,W;V^{(l)},U^{(l)}) = \sum_{M\in\Gamma_n}\left(\rho_{k,l}(Z-\overline{W})^{-1}(x\,{}^t y)^l\right)|_{k,l}M,$$

where we regard $(x\,{}^t y)^l$ as a $V^{(l)}\cdot U^{(l)}$-valued constant function.

Proposition 3.1. Let $m = \dim S_{k,l,n}(V^{(l)})$ and $f_1,\ldots,f_m$ be orthonormal basis of $S_{k,l,n}(V^{(l)})$. Then,

$$P_{k,l,n}(Z,W;V^{(l)},U^{(l)}) = C_{k,l,n}\sum_{j=1}^{m} f_j(Z)\cdot\sigma(\overline{f_j(W)}) \tag{3.1}$$

where

$$C_{k,l,n} = 2^{n(n-k+1)-l+1}i^{nk+l}\frac{\pi^{n(n+1)/2}}{k+l-1}\prod_{j=1}^{n-1}\frac{\Gamma(2k-2n+2j-1)(2k-n+j-2)^{[l]}}{(k-n-1+j)\Gamma(2k+j+l-n-1)}$$

Proof. The equation (3.1) is equivalent to

$$(f(Z),P_{k,l,n}(Z,W;V^{(l)},U^{(l)}))_{k,l} = C_{k,l,n}\sigma(f(W))$$

for all $f\in S_{k,l,n}(V^{(l)})$. Let S_n be the generalized unit circle of degree n:

$$S_n = \{S={}^tS\in M(n,\mathrm{C}) \mid E-S\overline{S}>0\}.$$

Then the similar computation to Klingen [6, Sect.1] gives

$$(f(Z),P_{k,l,n}(Z,W;V^{(l)},U^{(l)}))_{k,l}$$
$$= 2^{n(n-k+1)-l+1}i^{nk+l}\rho_{k,l}(\sqrt{\mathrm{Im}\,W}^{-1})\psi_{k-n-1,l,n}\rho_{k,l}(\sqrt{\mathrm{Im}\,W})\sigma(f(W))$$

where

$$\psi_{a,l,n} = \int_{S_n}\rho_{a,l}(E-S\overline{S})\,dS.$$

Changing variable S by ${}^tUSU$, we see

$$\psi_{a,l,n}=\rho_{a,l}(U^{-1})\psi_{a,l,n}\rho_{a,l}(U)$$

for any unitary matrix U. Since $\rho_{a,l}$ is an irreducible representation of $U(n,\mathrm{C})$, the operator $\psi_{a,l,n}$ is a homothety by the Schur's lemma. That is, there exists a constant $c_{a,l,n}$ satisfying $\psi_{a,l,n}$

$= c_{a,l,n}\mathrm{Id}$. Hence the proposition follows from

$$c_{a,l,n} = \frac{\pi^{n(n+1)/2}}{a+n+l}\prod_{j=1}^{n-1}\frac{\Gamma(2a+2j+1)(n+j+2a)^{[l]}}{(a+j)\Gamma(l+n+j+2a+1)}. \tag{3.2}$$

We compute $c_{a,l,n}$.

$$\begin{aligned} c_{a,l,n} &= (\psi_{a,l,n}x_1^l, x_1^l) \\ &= \int_{S_n} \det(E-S\overline{S})^a((E-S\overline{S})[x_1])^l dS \end{aligned}$$

We set $S = \begin{pmatrix} S_1 & {}^tv \\ v & z \end{pmatrix}$. By Hua[5, Sect.2.3], especially by Theorem 2.3.2 there,

$$\begin{aligned} c_{a,l,n} &= \frac{\pi}{a+1}\int_{I-S_1\overline{S_1}-{}^tv\overline{v}>0} \frac{\det(E-S_1\overline{S_1}-{}^tv\overline{v})^a((E-S_1\overline{S_1}-{}^tv\overline{v})[x_1])^l}{(1+\overline{v}(E-S_1\overline{S_1}-{}^tv\overline{v})^{-1}\,{}^tv)^{a+2}}\, dvdS_1 \\ &= \frac{\pi}{a+1}\int_{E-S_1\overline{S_1}>0}\det(E-S_1\overline{S_1})^{a+1}\int_{1-\overline{u}\,{}^tu>0}(1-\overline{u}\,{}^tu)^{2a+2}(\xi_1(E-{}^tu\overline{u})\,\overline{{}^t\xi_1})^l dudS_1, \end{aligned}$$

where $\xi_1 = \sqrt{E-S_1\overline{S_1}}x_1$. Put

$$\varphi_{a,l,n} = \int_{\substack{1-\overline{u}\,{}^tu>0 \\ u\in\mathbf{C}^n}} (1-\overline{u}\,{}^tu)^a \mathrm{Sym}^l(E-{}^tu\overline{u})\,du,$$

Using Schur's lemma again, there exist a constant $d_{a,l,n}$ satisfying $\varphi_{a,l,n} = d_{a,l,n}\mathrm{Id}$. Then,

$$\begin{aligned} &\int_{1-\overline{u}\,{}^tu>0}(1-\overline{u}\,{}^tu)^{2a+2}(\xi_1(E-{}^tu\overline{u})\,\overline{{}^t\xi_1})^l du \\ &= (\varphi_{2a+2,l,n-1}\xi_1^l, \xi_1^l) = d_{2a+2,l,n-1}(\xi_1,\xi_1)^l \\ &= d_{2a+2,l,n-1}((E-S_1\overline{S_1})[x_1])^l \end{aligned}$$

Therefore,

$$c_{a,l,n} = \frac{\pi}{a+1} c_{n-1,a+1,l}\, d_{2a+2,l,n-1}. \tag{3.3}$$

The value of $d_{a,l,n}$ is calculated as follows:

$$\begin{aligned} d_{a,l,n} &= (\varphi_{a,l,n} x_1, x_1) \\ &= \int_{\substack{1-\bar{u}\,{}^t u>0 \\ u\in \mathbf{C}^n}} (1-\bar{u}\,{}^t u)^a ((E-{}^t u\bar{u})[x_1])^l du \\ &= \int_{1-\sum_{j=1}^{2n} t_j^2>0} (1-\sum_{j=1}^{2n} t_j^2)^a (1-t_1^2-t_2^2)^l dt_1 \cdots dt_{2n} \\ &= \pi^n \frac{\Gamma(a+1)}{\Gamma(a+l+n+1)} (n+a)^{[l]}. \end{aligned} \tag{3.4}$$

By Hua[5, (2.2.6)],

$$c_{a,l,1} = \frac{\pi}{a+l+1}. \tag{3.5}$$

Summing up (3.3)-(3.5), we obtain (3.2). □

4. The Pullback Formula

In this section, we prove a vector valued version of the Garrett's Pullback formula. Let p and q be positive integers. To keep notation simple, we put

$$x_A = (x_1, \ldots, x_{p-r}),$$

$$x_B = (x_{p-r+1}, \ldots, x_p),$$

$$x_C = (x_{p+1}, \ldots, x_{p+q-r}),$$

$$x_D = (x_{p+q-r+1}, \ldots, x_{p+q})$$

and

$$V_{AB} = \mathrm{C}x_1 \oplus \ldots \oplus \mathrm{C}x_p,$$

$$V_B = \mathrm{C}x_{p-r+1} \oplus \ldots \oplus \mathrm{C}x_p,$$

$$V_{CD} = \mathrm{C}x_p \oplus \ldots \oplus \mathrm{C}x_{p+q},$$

$$V_D = \mathrm{C}x_{p+q-r+1} \oplus \ldots \oplus \mathrm{C}x_{p+q}$$

for an integer r with $0 \le r \le \min(p,q)$. Let σ be an isomorphism from $V_B^{(l)}$ to $V_D^{(l)}$ induced from $\sigma(x_{p-r+j}) = x_{p+q-r+j}$. For $f \in C^\infty(H_r, V_B^{(l)})$ we define $\sigma(f)$ by $(\sigma(f))(z) = \sigma(f(z))$. Let $P_{n,r}$ be the subgroup of Γ_n consisting of an element whose entries in last $n+r$ rows and first $n-r$ columns vanish. The Siegel's Eisenstein series $E_k^n(Z)$ of weight k and of degree n is

$$E_k^n(Z) = \sum_{g \in P_{n,0} \backslash \Gamma_n} (1 \mid_{k,0} g)(Z).$$

For $k \ge n+1$, this converges absolutely and uniformly on any compact set in H_n. Let U and V be any representation space of $\rho_{k,l,r}$ and $\rho_{k,l,n}$, respectively. Assume $U \subset V$ and

$$\rho_{k,l,n}\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} u = \det A^k \rho_{k,l,r}(D) u$$

for all $A \in GL(n-r, \mathrm{C})$, $D \in GL(r, \mathrm{C})$ and $u \in U$. For such a pair (U, V), we define the Klingen type Eisenstein series $E(f, V) \in M_{k,l,n}(V)$ attached to $f \in S_{k,l,r}(U)$ by

$$E(f,V)(z) = \sum_{g \in P_{n,r} \backslash \Gamma_n} ((f \mathrm{pr}_r^n \mid_{k,l} g)(z).$$

Here $\mathrm{pr}: H_n \to H_r$ is a projection defined by

$$\mathrm{pr}_r^n \begin{pmatrix} * & * \\ * & z \end{pmatrix} = z$$

where z is of size r.

Lemma 4.1. Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ and $Z \in H_n$. Let $k>0$ and $l>0$ be integers. Then

$$L^{(l)}(\det(CZ+D)^{-k}) = \alpha_{k,l}(\det(CZ+D)^{-k})((0\ \ 0\ \ x_C\ \ x_D)(CZ+D)^{-1}\,{}^tC\,{}^t(x_A\ \ x_B\ \ 0\ \ 0))^l$$

where

$$\alpha_{k,l} = (-1)^l \frac{(2k-2)^{[l]}}{l!(k-1)^{[l]}}.$$

Especially, let M be a symmetric matrix of size $0 \leq r \leq \min(p,q)$ and

$$g_{\tilde{M}} = \begin{pmatrix} E & 0 \\ 0 & C \end{pmatrix} \tag{4.1}$$

where $C = \begin{pmatrix} 0 & \tilde{M} \\ {}^t\tilde{M} & 0 \end{pmatrix}$ with $\tilde{M} = \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}$. (We understand that $C = 0$ for $r=0$.) Then,

$$L^{(l)}\left(1\,|_{k,0}\,g_{\tilde{M}}\right)(Z,W) = \begin{cases} \alpha_{k,l}\rho_{k,l}(E-Mw_3Mz_3)(x_B\,{}^t(x_DM))^l & (r>0), \\ 0 & (r=0), \end{cases}$$

where $Z = \begin{pmatrix} z_1 & {}^tz_2 \\ z_2 & z_3 \end{pmatrix} \in H_p$ and $W = \begin{pmatrix} w_1 & {}^tw_2 \\ w_2 & w_3 \end{pmatrix} \in H_q$.

Proof. Straightforward (but long). □

Let $\tilde{S}$ be the symmetric square operator acting on $S_{k,l,n}(V^{(l)})$, which is defined by

$$\tilde{S}f = \sum_M \sum_{g \in \Gamma_n \backslash \Gamma_n \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} \Gamma_n} f\,|_{k,l}\,g$$

where M runs over all non-singular integral matrices of size n in elementary form. By Garrett[4, Prop. in Sect. 4], a common eigenfunction of all Hecke operator is an eigenfunction of $\tilde{S}$. Moreover, by [1, (6)], its eigenvalue $\Lambda(f)$ is

$$\zeta(k)^{-1}\prod_{i=1}^{n}\zeta(2k-2i)^{-1}D_f(k-n)$$

where $\zeta(s)$ is the Riemann zeta function and $D_f(s)$ is the standard L-function of f. For simplicity we put $N_{k,l,n} = \dim S_{k,l,n}(V^{(l)})$ for $n \geq 1$.

Proposition 4.2. Let p, $q>1$ be integers and $Z \in H_p$, $W \in H_q$. Let $k \geq p+q+1$ and $l \geq 2$ be even integers. For $1 \leq r \leq \min(p,q)$, let $\{f_{j,r}\}_{1\leq j\leq N_{k,l,r}}$ be an orthonormal basis of common eigenfunction. Then,

$$(L^{(l)}E_k^{p+q})(Z,W)$$

$$= \alpha_{k,l}\sum_{r=1}^{\min(p,q)} C_{k,l,r}\sum_{j=1}^{N_{k,l,n}} \Lambda(f_{j,r})E(f_{j,r},V_{AB}^{(l)})(Z)E(\sigma\theta(f_{j,r}),V_{CD}^{(l)})(W)$$

where θ is an operator defined by $(\theta f)(z) = \overline{f(-\bar{z})}$.

Proof. Let $g_{\hat{M}}$ be as in (4.1). By the same computation as in Garrett[4, Sect. 5]

$$\sum_{g_0'\in\Gamma_r} L^{(l)}(1\,|_{k,0}g_{\hat{M}})\,|_{k,l}g_0'(Z,W)$$

$$= \alpha_{k,l}\det M^{-k}\left(\sum_{g\in\Gamma_r}\rho_{k,l,r}(z_3+w_3)(x_B\,{}^tx_D)^l\,|_{k,l,p}g\right)|_{k,l,q}\hat{M}$$

where $\hat{M} = \begin{pmatrix} M & 0 \\ 0 & M^{-1}\end{pmatrix}$. By Proposition 3.1, this is

$$\alpha_{k,l}\det M^{-k}P_{k,l,r}(z_3,-\overline{w_3};V_B^{(l)},V_D^{(l)})\,|_{k,l,q}\hat{M}$$

$$= \alpha_{k,l} \det M^{-k} C_{k,l,n} \sum_{j=1}^{N_{k,l,r}} f_{j,r}(z_3)(\sigma\theta(f_{j,r}) |_{k,l,q} \hat{M})(w_3)$$

Hence, as in [4, Sect. 5], we have

$$(L^{(l)} E_k^{p+q})(Z, W)$$

$$= \alpha_{k,l} \sum_{r=1}^{\min(p,q)} C_{k,l,n} \sum_{j=1}^{N_{k,l,r}} \sum_{g_0'' \in P_{p,r} \backslash \Gamma_p} (f_{j,r} \mathrm{pr}_r^p |_{k,l} g_0'')(Z)$$

$$\times \sum_{g_1'' \in P_{q,r} \backslash \Gamma_q} ((\tilde{S} \sigma\theta(f_{j,r})) \mathrm{pr}_r^q |_{k,l} g_1'')(W)$$

$$= \alpha_{k,l} \sum_{r=1}^{\min(p,q)} C_{k,l,n} \sum_{j=1}^{N_{k,l,r}} \Lambda(f_{j,r}) E(f_{j,r}, V_{AB}^{(l)})(Z) E(\sigma\theta(f_{j,r}), V_{CD}^{(l)})(W)$$

□

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