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Discrete series for semisimple symmetric spaces

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The aim of this note is to explain the essence of the theory of discrete series for semisimple symmetric spaces $X = G/H$ in [7], [6] and [3].

Let $\mathfrak{g}$ be a real semisimple Lie algebra and $\mathfrak{g}_c$ the complexification of $\mathfrak{g}$. Let σ be an involution ($\sigma^2 = id.$) of $\mathfrak{g}$ and θ a Cartan involution of $\mathfrak{g}$ such that $\sigma\theta = \theta\sigma$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be $+1, -1$-eigenspace decompositions for σ and θ, respectively. Then

$$\mathfrak{g} = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{s} \cap \mathfrak{h}) \oplus (\mathfrak{s} \cap \mathfrak{q}).$$

Put $\mathfrak{h}^d = (\mathfrak{k} \cap \mathfrak{h}) \oplus i(\mathfrak{k} \cap \mathfrak{q})$, $\mathfrak{k}^d = (\mathfrak{k} \cap \mathfrak{h}) \oplus i(\mathfrak{s} \cap \mathfrak{h})$ and $\mathfrak{g}^d = \mathfrak{h}^d + \mathfrak{k}^d + (\mathfrak{s} \cap \mathfrak{q})$. Then $(\mathfrak{h}^d)_c = \mathfrak{k}_c$, $(\mathfrak{k}^d)_c = \mathfrak{h}_c$ and $(\mathfrak{g}^d)_c = \mathfrak{g}_c$.

Let G_c be a connected Lie group with Lie algebra $\mathfrak{g}_c$. Let G, K, H, G^d, K^d, H^d, K_c and H_c be the analytic subgroups of G_c for $\mathfrak{g}, \mathfrak{k}, \mathfrak{h}, \mathfrak{g}^d, \mathfrak{k}^d, \mathfrak{h}^d, \mathfrak{k}_c$ and $\mathfrak{h}_c$, respectively. Then $X = G/H$ is called a *semisimple symmetric space* and $X^d = G^d/K^d$ is a Riemannian symmetric space of noncompact type. Both of X and X^d are "real forms" of a *complex symmetric space* $X_c = G_c/H_c$. (Remark. We don't have to assume that σ lifts to G_c in the following.)

Example 1. Let $G_c = SL(2, \mathbb{C})$, $G = SL(2, \mathbb{R})$, $\theta g = {}^t g^{-1}$ and

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \text{ for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_c.$$

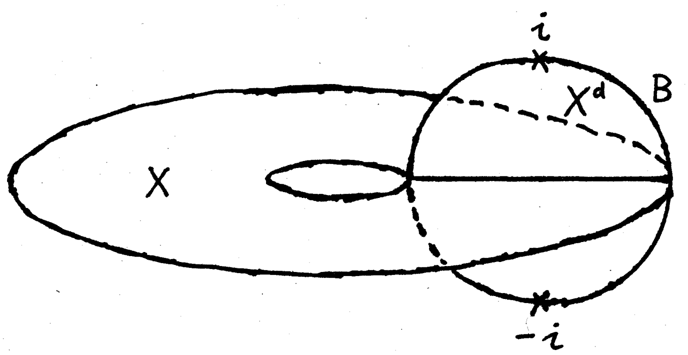
Then

$$K = SO(2), H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R}_{>0} \right\},$$

$$K^d = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid |a| = 1 \right\}, H^d = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a \in \mathbb{R}_{>0}, b \in i\mathbb{R}, a^2 + b^2 = 1 \right\},$$

$$G^d = SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C}, a\bar{a} - b\bar{b} = 1 \right\},$$

X^d is identified with the unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$ with the G^d-action



$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} z = \frac{az+b}{\bar{b}z+\bar{a}} \text{ for } \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in G^d \text{ and } z \in X^d,$$

$X = G/H$ intersects with X^d on the real axis, and the H^d-orbits on the boundary $B = \{z \in \mathbf{C} \mid |z| = 1\}$ of X^d are $\{i\}, \{-i\}, \{z \in B \mid \mathrm{Re} z > 0\}$ and $\{z \in B \mid \mathrm{Re} z < 0\}$.

Let $\underset{\sim}{\mathcal{A}}_K(X)$ (resp. $\underset{\sim}{\mathcal{A}}_{H^d}(X^d)$) be the space of K-finite (resp. K_c-finite) analytic functions on X (resp. X^d). Here "a function f on X^d is K_c-finite" implies that it is H^d-finite and the representation of H^d on the space spanned by $H^d f$ lifts to a holomorphic representation of K_c. Then the analytic continuation in X_c gives an isomorphism $f \mapsto f^\eta$ of $\underset{\sim}{\mathcal{A}}_K(X)$ onto $\underset{\sim}{\mathcal{A}}_{H^d}(X^d)$ which commutes with the left $\mathfrak{g}_c$-action and the $\mathbf{D}(X)$-action ([1]). Here $\mathbf{D}(X)$ is the ring of G-invariant differential operators on X. By the analytic continuation, $\mathbf{D}(X)$ is identified with $\mathbf{D}(X^d)$ the ring of G^d-invariant differential operators on X^d.

Let $\mathfrak{a}^d$ be a maximal abelian subspace of $\mathfrak{s}^d = i(\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{s} \cap \mathfrak{q})$ such that $\mathfrak{a} = \mathfrak{a}^d \cap \mathfrak{s}$ is maximal abelian in $\mathfrak{s} \cap \mathfrak{q}$. Let Σ^+ be a positive system of the root system $\Sigma(\mathfrak{a}^d)$ such that $\langle \Sigma^+, Y \rangle \subset \mathbf{R}_{\geq 0}$ for a generic Y in $\mathfrak{a}$. Put $A = \exp \mathfrak{a}$ and define a closed subset $A^+ = \{a \in A \mid a^\alpha \geq 1 \text{ for all } \alpha \in \Sigma^+\}$ of A. Let P be a minimal parabolic subgroup of G^d defined by

$$P = P(\mathfrak{a}^d, \Sigma^+) = M^d A^d N^d$$

where $M^d = Z_{K^d}(A^d)$ (the centralizer of A^d in K^d), $A^d = \exp \mathfrak{a}^d$, $\mathfrak{n}^d = \sum_{\alpha \in \Sigma^+} \mathfrak{g}^d(\mathfrak{a}^d; \alpha)$ and $N^d = \exp \mathfrak{n}^d$. Put $J = N_{K^d}(A)/N_{K \cap H}(A) Z_{K^d}(A) = N_K(A)/N_{K \cap H}(A) Z_K(A)$ where $N_*(**)$ is the normalizer of $**$ in $*$.

Proposition 1 (c.f. [1]). $G = K(\bigcup_{m \in J} m A^+ m^{-1}) H$

Proposition 2 ([4]). *$\{H^d m P \mid m \in J\}$ is the set of open H^d-orbits on G^d/P.*

Let λ be a complex linear form on $\mathfrak{a}_c^d$ and let $\underset{\sim}{\mathcal{B}}(G^d/P; L_\lambda)$ be the space of hyperfunctions on G^d satisfying $f(xman) = a^{\lambda-\rho} f(x)$ for $x \in G^d$, $m \in M^d$, $a \in A^d$ and $n \in N^d$ where $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ ($m_\alpha = \dim \mathfrak{g}^d(\mathfrak{a}^d; \alpha)$). Suppose that $\mathrm{Re}\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Sigma^+$. Then the Poisson transform p_λ defined by

$$(p_\lambda f)(x) = \int_{K^d} f(xk)dk = \int_{K^d} h(x^{-1}k)^{-\lambda-\rho} f(k)dk$$

(where $h : G^d \to A^d$ is the projection with respect to the Iwasawa decomposition $G^d = K^d A^d N^d$) gives a G^d-isomorphism of $\underset{\sim}{\mathcal{B}}(G^d/P; L_\lambda)$ onto $\underset{\sim}{\mathcal{A}}(X^d)_\lambda = \{f \in \underset{\sim}{\mathcal{A}}(X^d) \mid Df = \chi_\lambda(D) f \text{ for } D \in \mathbf{D}(X^d)\}$ ([2]). Here χ_λ is the character of $\mathbf{D}(X^d)$ parametrized by λ. Note that $\chi_\lambda = \chi_\nu \iff \nu \in W\lambda$ where $W = N_{K^d}(A^d)/Z_{K^d}(A^d)$ is the Weyl group of $\Sigma(\mathfrak{a}^d)$.

Let g be an H^d-finite element in $\underset{\sim}{\mathcal{B}}(G^d/P; L_\lambda)$. Then $V = \mathrm{supp} g$ is a closed H^d-invariant subset of G^d/P.

Example 2([1]). Let $V = H^d x_0 P = (K \cap H) x_0 P$ be a closed H^d-orbit on G^d/P. Define a distribution T on K^d/M^d by

$$\langle T, \varphi \rangle = \int_{K \cap H} \varphi(kx_0) dk \text{ for } \varphi \in C^\infty(K^d/M^d)$$

The distribution T is identified with an element T_λ of $B(G^d/P; L_\lambda)$ by the inclusion $K^d \hookrightarrow G^d$ and T_λ becomes H^d-finite under some condition on λ. Flensted-Jensen defined generating functions $\psi_\lambda \in \underset{\sim}{A}_K(X)_\lambda = \{f \in \underset{\sim}{A}_K(X) \mid Df = \chi_\lambda(D)f \text{ for } D \in \mathbf{D}(X)\}$ of discrete series for X by

$$\psi_\lambda^\eta(x) = (p_\lambda T_\lambda)(x) = \int_{K \cap H} h(x^{-1} k x_0) dk.$$

(Discrete series for X are the representations of G realized in subspaces of $L^2(X)_\lambda = \{f \in L^2(X) \mid Df = \chi_\lambda(D)f \text{ for } D \in \mathbf{D}(X)\}$ for some λ).

For V and $m \in J$, define a subset $W_{V,m} = \{w \in W \mid V(Pw^{-1}P)^{cl} \supset H^d mP$ and $V(Pv^{-1}P)^{cl} \not\supset H^d mP$ if $(Pv^{-1}P)^{cl} \subsetneq (Pw^{-1}P)^{cl}\}$ of W. Put $S_{V,m,\lambda} = W_{V,m}\lambda|_A$. Assume $\mathrm{Re}\langle \lambda, \alpha \rangle > 0$ ($\alpha \in \Sigma^+$) for simplicity in the following.

Theorem ([6]). *Let $m \in J$, $f \in \underset{\sim}{A}_K(X)_\lambda$ and put $V = V_f = \mathrm{supp}\ p_\lambda^{-1}(f^\eta)$. Then there exist nonzero analytic functions f_μ on K for all $\mu \in S_{V,m,\lambda}$ such that*

$$f(kmam^{-1}H) = \sum_{\mu \in S_{V,m,\lambda}} f_\mu(k) a^{\mu-\rho} + o\Big(\sum_{\mu \in S_{V,m,\lambda}} |a^{\mu-\rho}|\Big)$$

($k \in K$) when $a^\alpha \to +\infty$ for all $\alpha \in \Sigma^+|_{\mathfrak{a}} \setminus \{0\}$.

Remark. Above formula gives the asymptotic behavior of f at the minimal boundaries of X. But we can see also the asymptotic behavior at other boundaries from this formula since we have expansions of f at these boundaries and the boundary values are analytic([6]).

Corollary ([6]). *Let $f \in \underset{\sim}{A}_K(X)_\lambda$. Then $f \in L^2(X) \iff$ (P) $|a^\mu| < 1$ for any $\mu \in \bigcup_{m \in J} S_{V_f,m,\lambda}$ and $a \in A^+ \setminus \{1\}$.*

Lemma ([7] Lemma 7 + [3] Lemma 1.2). (P) $\iff$ (i) $\mathrm{rank} X = \mathrm{rank}(K/K \cap H)$ *and* (ii) $V_f \subset$ *the union of closed H^d-orbits on G^d/P.* ((ii) $\iff$ $\dim V_f$ is the smallest.)

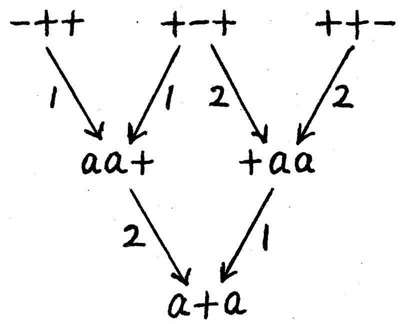
Remark. By the above corollary and the lemma, we don't need case-by-case checkings in [7] p.361-p.377.

Example 3. Let $X = G/H = SU(2,1) \times SU(2,1)/diagonal \cong SU(2,1)$. Then $G^d \cong SL(3, \mathbb{C})$, $H^d \cong (GL(1, \mathbb{C}) \times GL(2, \mathbb{C})) \cap G^d$ and $\mathfrak{s}^d \cong \{$*hermitian matrices in* $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})\}$. Put

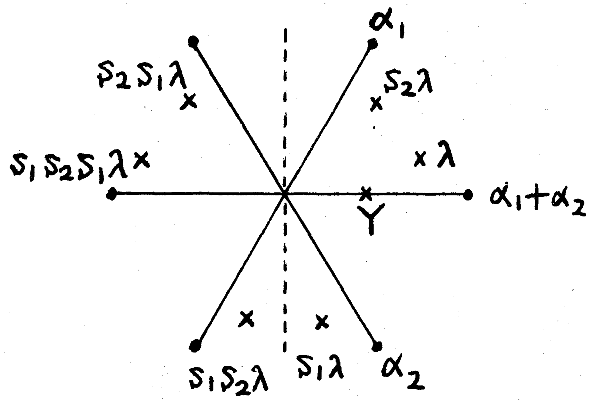
$$Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $\mathfrak{a} = \mathbb{R}Y$ is a maximal abelian subspace of $\mathfrak{s}^d \cap \mathfrak{q}^d$ and $\mathfrak{a}^d = \mathfrak{z}_{\mathfrak{s}^d}(Y)$ is a maximal abelian subspace of $\mathfrak{s}^d$. Put $\Sigma^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} = \{\alpha \in \Sigma(\mathfrak{a}^d) \mid \alpha(Y) > 0\}$ and define a minimal parabolic subgroup P of G^d from $\mathfrak{a}^d$ and Σ^+. Let s_i denote the reflection with respect to $\alpha_i (i = 1, 2)$.

There are six H^d-orbits $V_1 = -++, V_2 = +-+, V_3 = ++-$, $V_4 =$aa+, $V_5 =$+aa and $V_6 = H^dP =$a+a on the flag manifold G^d/P where V_1, V_2 and V_3 are closed and V_6 is open ([5]). We can see that $W_{V_1,1} = \{s_2s_1\}, W_{V_2,1} = \{s_2s_1, s_1s_2\}$ and $W_{V_3,1} = \{s_1s_2\}$ from the diagram of the orbit structure. (The diagram implies that $V_1(Ps_1P)^{cl} = V_1 \cup V_2 \cup V_4$, for instance.) We get easily that $\mathrm{Re}(s_2s_1\lambda)(Y) < 0$ and $\mathrm{Re}(s_1s_2\lambda)(Y) < 0$ from the assumption $\mathrm{Re}\langle\lambda, \alpha_i\rangle > 0 (i = 1, 2)$. Hence the property (P) holds for V_1, V_2 and V_3. On the other hand, let V be a closed H^d-invariant subset of G^d/P such that $V \not\subset V_1 \cup V_2 \cup V_3$. Then $V \supset V_4$ or $V \supset V_5$ and therefore $W_{V,1} \ni s_2$ or $W_{V,1} \ni s_1$. Since $\mathrm{Re}(s_i\lambda)(Y) > 0$, the property (P) does not hold. The discrete series coming from V_1 and V_3 are the holomorphic and anti-holomorphic discrete series for $X = SU(2, 1)$ and the one coming from V_2 is the other one.



H^d- orbit structure on G^d/P



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