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Asymptotic Analyses for the Emden-Fowler Equation $-\Delta u = \lambda e^u$

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§1. Introduction.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the smooth boundary $\partial\Omega$, and λ be a positive constant. For the classical solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of the Emden-Fowler equation

(1.1) $$-\Delta u = \lambda e^u \quad (\text{in } \Omega), \qquad u = 0 \quad (\text{on } \partial\Omega)$$

we have proven the following theorem in [3], where

(1.2) $$\Sigma = \int_\Omega \lambda e^u dx.$$

Theorem 1. As $\lambda \downarrow 0$, the values $\{\Sigma\}$ accumulate to $8\pi m$ for some $m = 0, 1, \cdots, +\infty$. The solutions $\{u\}$ behave as follows:

(a) If $m = 0$, then $\|u\|_{L^\infty(\Omega)} \to 0$

(b) If $0 < m < +\infty$, then there exists a set of m-points $\mathcal{S} = \{x_1^*, \cdots, x_m^*\} \subset \Omega$ such that $\|u\|_{L^\infty_{loc}(\overline{\Omega}\setminus\mathcal{S})} \in O(1)$ and $u|_{\mathcal{S}} \to +\infty$.

(c) If $m = +\infty$, then $u(x) \to +\infty$ $(x \in \Omega)$.

In the case (b), the limiting function of $\{u\}$, the singular limit, is given as

(1.3) $$u_0(x) = 8\pi \sum_{j=1}^{m} G(x, x_j^*)$$

and the blow-up points $\mathcal{S}$ is located as

$$\frac{1}{2}\nabla R(x_j^*) + \sum_{\ell \neq j} \nabla_x G(x_\ell^*, x_j^*) = 0 \quad (1 \leq j \leq m),$$

where $G(x, y)$ denotes the Green function of $-\Delta$ in Ω under the homogeneous Dirichlet boundary condition and $R(x) = [G(x, y) + \frac{1}{2\pi} \times \log|x-y|]_{y=x}$ is the Robin function. In other words, $(x_1^*, \cdots, x_m^*) \in \Omega \times \cdots \times \Omega$ $(x_\ell \neq x_{\ell'}$ for $\ell \neq \ell')$ is a critical point of the function

(1.4) $\quad K(x_1, \cdots, x_m) = \frac{1}{2}\sum_{j=1}^{m} R(x_j) + \sum_{\ell < \ell'} G(x_\ell, x_{\ell'})$

in $\Omega \times \cdots \times \Omega$.

The purpose of the present article is to study the inverse problem. The following theorem can be proven:

<u>Theorem 2</u>. Suppose that Ω is simply connected, and let $(x_1^*, \cdots, x_m^*)$ be a non-degenerate critical point of $K(x_1, \cdots, x_m)$. Then there exists a family of classical solutions $\{u\}$ of (1.1) with sufficient small $\lambda > 0$ such that

(1.5) $\quad u(x) \longrightarrow u_0(x) \quad (x \in \Omega)$

as $\lambda \downarrow 0$, where $u_0(x)$ is the function in (1.3).

Thus, we have almost classified the solutions of (1) when $\lambda > 0$ is sufficiently small. As for the global bifurcation diagram, see

[4].

§2. Outline of Proof

Theorem 2 with $m=1$ has been proven for a generic case by V. H. Weston and J. L. Moseley by means of a Newton's iteration scheme ([5], [2]). What we shall do is that their fixed point equation can be solved by a simple iteration scheme without Newton's one, reducing their assumptions, and that the same argument on a Riemannian surface with m-leaves produces the multi-point blow up solutions.

First, we observe that the singular limit $u_0(x)$ in (1.3) with $m=1$, is related to the conformal mapping $h : \Omega \longrightarrow D \equiv \{|z|<1\}$ satisfying $h(x_1^*)=0$ when $\Omega \subset \mathbb{R}^2$ is simply connected. In fact we have

$$e^{-u_0} = |h|^2. \tag{2.1}$$

The condition $\nabla R(x_j^*)=0$ is equivalent to $h''(x_j^*)=0$. Here, we deduce the equation satisfied the function $v=e^{-u}$, that is,

$$v\Delta v - |\nabla v|^2 = \frac{\lambda}{2} \quad (\text{in } \Omega), \qquad v=1 \quad (\text{on } \partial\Omega). \tag{2.2}$$

We want to construct a family of non-negative solutions $\{v\}$ for small $\lambda>0$ so that

$$v \longrightarrow |h|^2 \quad \text{as } \lambda \downarrow 0. \tag{2.3}$$

Introducing the inverse mapping $g = h^{-1} : D \to \Omega$, we pull-back those relations into D so that $V = g^{*}v$ satisfies

(2.4) $$V\Delta V - |\nabla V|^2 = \frac{\lambda}{2}|g'|^2 \quad (\text{in } D), \qquad V = 1 \quad (\text{on } \partial D)$$

with

(2.5) $$V \to |z|^2 \qquad \text{as } \lambda \downarrow 0.$$

We note the relation

(2.6) $$g''(0) = 0$$

implied by $h''(x_1^*) = 0$.

The left-hand side of (2.4) is a quadratic form of V, which we write as $Q(V)$ to define

(2.7) $$Q\{V, W\} = \frac{1}{2}(V\Delta W + W\Delta V) - \nabla V \cdot \nabla W.$$

A calculation yields

(2.8) $$Q\{|p|^2, |g|^2\} = 2|\omega_z(p, g)|^2$$

for holomorphic functions $p(z)$ and $g(z)$, where $\omega_z(p, g)$ denotes the Wronskian;

(2.9) $$\omega_z(p, g) = pg' - p'g.$$

In particular we have $Q(|p|^2) = 0$.

Now we can introduce the integral for the first relation of (2.4), which was essentially discovered by J. Liouville [1]:

(2.10) $$V = |z/G|^2 + \frac{\lambda}{8}|M/G|^2,$$

where $G(z)$ and $M(z)$ are holomorphic functions. From (2.8) we have that

$$Q(V) = \frac{\lambda}{4} Q\{|z/G|^2, |M/G|^2\} = \frac{\lambda}{2} |\omega_z(z/G, M/G)|^2$$
$$= \frac{\lambda}{2} |\omega_z(z,M)/G^2|^2.$$

We have only to solve that

(2.11) $\omega_z(z, M) = g' G^2$ (in D) with $|G|^2 = 1 + \frac{\lambda}{8}|M|^2$ (on ∂D).

Given $K(z)$, the solvability of

(2.12) $\omega_z(z, M) = K$

is equivalent to $K'(0) = 0$. Then the solution $M(z)$ is given as

(2.13) $M = d(K) + az,$

where $a \in \mathbb{C}$ denotes the integral constant and $d(K) = \sum_{n \neq 1} \frac{k_n}{n-1} z^n$ for $K(z) = \sum k_n z^n$. The problem (2.11) is reduced to

(2.14) $|G|^2 = 1 + \frac{\lambda}{8} |d(g'G^2) + az|^2$ on ∂D with $G'(0) = 0.$

The relation (2.5) is realized by

(2.15) $G \longrightarrow 1$ as $\lambda \downarrow 0.$

Here, we put $G = 1 + \lambda H$ to deduce

(2.16) $H + \overline{H} = \frac{1}{8}|d(g') + az|^2 + \lambda \Phi(H, a, \lambda)$ on ∂D with $H'(0) = 0,$

where

(2.17) $\Phi(H, a, \lambda) = -|H|^2 + \frac{1}{8}\{(d(g') + az)\overline{(2d(g'H) + \lambda d(g'H^2))}$

$$+ \overline{(d(g')+az)}\,(2d(g'H) + \lambda d(g'H^2)) + |2d(g'H) + \lambda d(g'H^2)|^2 \}.$$

Putting $d(g') = C_0 + z^2 I_0(z)$ with $C_0 \in \mathbb{C}$, we have

$$|d(g') + az|^2 = |a|^2 + 2\,\mathrm{Re}\,\{(a\overline{C_0} + \overline{a}\, I_0(z))z\} + |d(g')|^2 \quad \text{if } |z| = 1.$$

Therefore, the first relation of (2.14) is equivalent to

$$(2.18) \quad 2\,\mathrm{Re}\,H = \frac{|a|^2}{8} + \frac{1}{4}\mathrm{Re}\,\{(a\overline{C_0} + \overline{a}\, I_0(z))z\} + \frac{1}{8}|d(g')|^2 + \lambda \Phi \quad (\text{on } \partial D).$$

Utilizing Schwarz's formula we have

$$(2.19) \quad H(z) = \frac{|a|^2}{16} + \frac{1}{8}\{(a\overline{C_0} + \overline{a}\, I_0(z))z\}$$

$$+ \frac{1}{2\pi\sqrt{-1}} \int_{\partial D} \left\{ \frac{|d(g')|^2}{8} + \lambda \Phi(H, a, \lambda) \right\} \left\{ \frac{1}{\zeta - z} + \frac{1}{2\zeta} \right\} d\zeta \quad \text{for } z \in D.$$

Hence $H'(0) = 0$, the second relation of (2.16), is realized by

$$(2.20) \quad a\overline{C_0} + \overline{a}\, I_0(0) = \varphi(H, a, \lambda),$$

where

$$(2.21) \quad \varphi(H, a, \lambda) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial D} \left\{ \frac{|d(g')|^2}{8} + \lambda \Phi(H, a, \lambda) \right\} \frac{d\zeta}{\zeta^2}.$$

For given holomorphic function $H(z)$ with $H'(0) = 0$, the equation (2.20) is a linear equation of $(a, \overline{a})$ if $\lambda = 0$. In the case of $|I_0(0)/C_0| \neq 1$, which is equivalent to

$$(2.22) \quad |g'''(0)/g'(0)| \neq 2,$$

the equation (2.20) is solved with respect to a if $\lambda = 0$, and so is true for small $\lambda > 0$ by the implicit function theorem.

Thus we obtain

(2.23) $\quad a = a(H, \lambda) \quad$ if $\quad 0 < \lambda < \lambda(\|H\|_{L^\infty(D)}) << 1$.

Substituting this into the right-hand side of (2.19), we get the fixed point equation

(2.24) $\quad H = N(H, \lambda)$.

Here, $N(H, \lambda)'(0) = 0$ follows from $H'(0) = 0$ as is imposed in (2.20). $H \equiv 0$ solves (2.24) when $\lambda = 0$.

The formal consideration given above will be realized if we can prepare appropriate function spaces. Let HL be the Hardy-Lebesgue class:

$$HL = \{f(z) \mid \text{holomorphic in } D \text{ with } \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < +\infty\}.$$

We put

$$HL_0 = \{f \in HL \mid f'(0) = 0\} \quad \text{and} \quad HL_0^1 = \{f \in HL_0 \mid f' \in HL\}.$$

The functional α defined before is an isomorphism from HL_0 onto HL_0^1. The inclusion $HL_0^1 \subset X = \{f(z) \mid$ holomorphic in D, continuous on $\overline{D}$, and $f'(0) = 0\}$ assures us of the nonlinear operator

$$N = N(\cdot, \lambda) : B \longrightarrow B$$

defined above, where B denotes the unit ball of the Banach space X, provided that $\lambda > 0$ is sufficiently small. Furthermore, then the mapping $N(\cdot, \lambda)$ is a contraction. Hence it has a fixed point.

We note that the condition (2.22) is equivalent to the non-degeneracy of the critical point x_1^* of the Robin function $R(x)$.

The multi-point blow-up solutions can be constructed in the following way. As before, we first observe that the singular limit $v_0(x)$ of (1.3) satisfies

(2.25) $$e^{-g^* v_0} = |w|^2,$$

where $w = \prod_{j=1}^{n} \frac{z-\delta_j}{1-\bar{\delta}_j z}$ $(|\delta_j|<1)$ is a finite Blaschke product and $x_j^* = g(\delta_j)$. In view of this we introduce the integral

(2.26) $$V = |w/G|^2 + \frac{\lambda}{8}|M/G|^2 \ (\text{in } D), \quad V = 1 \ (\text{on } \partial D)$$

for the first relation of (2.4). From

$$Q(V) = \frac{\lambda}{4} Q\{|w/G|^2, |M/G|^2\} = \frac{\lambda}{2}\left|W_z\left(\frac{w}{G}, \frac{M}{G}\right)\right|^2$$
$$= \frac{\lambda}{2}|W_z(w, M)/G^2|^2$$

is deduced that

(2.27) $$W_z(w, M) = g' G^2 \ (\text{in } D) \ \text{with } |G|^2 = 1 + \frac{\lambda}{8}|M|^2 \ (\text{on } \partial D)$$

Here, we introduce the Riemannian surface $\hat{D}$ so that the mapping $z \in D \longmapsto w \in \hat{D}$ is a homeomorphism. It is an m-covering of D, and a similar formula to that of Schwarz holds on it. The holomorphic function $g(z)$ in D induces that of $\hat{g} = \hat{g}(w)$ of w in $\hat{D}$ through the relation $g(z) = \hat{g}(w(z))$.

We can show that $(x_1^*, \cdots, x_m^*)$ is a non-degenerate critical point of K in (1.4) if and only if

(2.28) $\qquad \hat{g}''(0)=0 \quad$ and $\quad |\hat{g}'''(0)/\hat{g}'(0)| \neq 2.$

Then we can solve the equation

(2.29) $\qquad \omega_w(w, \hat{M}) = \hat{g}'\hat{G}^2 \ (\text{in } \hat{D}) \quad$ with $\quad |\hat{G}|^2 = 1 + \frac{\lambda}{8}|\hat{M}|^2 \ (\text{on } \partial\hat{D})$

to obtain holomorphic functions $\hat{G}(w)$ and $\hat{M}(w)$ of w in $\hat{D}$, satisfying

(2.30) $\qquad \hat{G}(w) \longrightarrow 1 \quad (\text{as } \lambda \downarrow 0)$

The holomorphic functions $G(z) = \hat{G}(w(z))$ and $M(z) = \hat{M}(w(z))$ satisfy (2.27). Thus the multi-point blow-up solutions have been constructed.

References

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