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The singularty on interpolation by rational spline functions

R.H.Wang
Inst. of Appl.Math.
Dalian Univ. of Tech.
Dalian, CHINA

T.Torii
Dept.of Infor. Eng.
Nagoya Univ.
Nagoya, JAPAN

Let $\mathbf{P}_n$ be the collection of all polynomials of degree n,and $\mathbf{R}_l^r$ the collection of all rational functions with the form $p(x)/q(x)$,where $p \in \mathbf{P}_r$, and $q \in \mathbf{P}_l$.

Denote by T the following partition of the interval $[a, b]$:

$$T: \quad a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

If a real function $R(x)$ definded on $[a, b]$ satisfies

1° $\quad R(x) \in \mathbf{R}_l^r$, in each interval $[x_j, x_{j+1}]$;

2° $\quad R(x) \in C^s[a, b]$,

then $R(x)$ is said to be a rational spline of type$-(r, l)^s$ whth respect to the partition T.

In this paper, we shell discuss the rational splines of type$-(2, 1)^1$,and type$-(2, 1)^2$ with the forms

$$R(x) = p_{1j}(x) + \frac{(x - x_j)(x - x_{j+1})}{q_{1j}(x)}, \quad x_j \leq x \leq x_{j+1}, \quad j = 0, \cdots, n-1, \tag{1}$$

where $p_{1j}(x)$ and $q_{1j}(x) \in \mathbf{P}_1$.

Suppose that the interpolation conditions are

$$\begin{cases} R(x_j) = y_j, & R'(x_j) = y_j', \\ R(x_{j+1}) = y_{j+1}, & R'(x_{j+1} = y_{j+1}'; \end{cases} \tag{2}$$

$$\begin{cases} R(x_j) = y_j, \qquad R(x_{j+1}) = y_{j+1}, \\ R'(x_j) = [f(x_{j-1}, x_j) + f(x_j, x_{j+1})]/2, \\ R'(x_{j+1}) = [f(x_j, x_{j+1}) + f(x_{j+1}, x_{j+2})]/2; \end{cases} \tag{3}$$

$$\begin{cases} R(x_j) = y_j, & R(x_{j+1}) = y_{j+1}, \\ R'(x_j) = y_j', & R''(x_j) = y_j'', \end{cases} \tag{4}$$

respectively ,where $f(x_j, x_{j+1})$ denotes the divided difference of the first degree, etc.

Denote by $R_{1j}(x)$ $R_{2j}(x)$, and $R_{3j}(x)$ the rational spline functions of satisfying the interpolation conditions (1)–(2),(1)–(3), and (1)–(4) respectively.

R.H. Wang and S.T. Wu([1][2]) have obtained the following rational piecewise functions which are satisfying the interpolation conditions (1)–(2),(1)–(3),and (1)–(4) respectively,

$$\begin{aligned} R_{1j}(x) &= y_j + f(x_j, x_{j+1})(x - x_j) \\ &+ \frac{(x - x_j)(x - x_{j+1})[y_j' - f(x_j, x_{j+1})][y_{j+1}' - f(x_j, x_{j+1})]}{(x - x_j)[y_j' - f(x_j, x_{j+1})] + (x - x_{j+1})[y_{j+1}' - f(x_j, x_{j+1})]}, \end{aligned} \tag{5}$$

$$\begin{aligned} R_{2j}(x) &= y_j + f(x_j, x_{j+1})(x - x_j)+ \\ &\{(x - x_j)(x - x_{j+1})[f(x_{j-1}, x_j) - f(x_j, x_{j+1})][f(x_{j+1}, x_{j+2}) \\ &- f(x_j, x_{j+1})]\}/\{2[f(x_{j-1}, x_j) + f(x_{j+1}, x_{j+2}) - 2f(x_j, x_{j+1})]x \\ &- 2[f(x_{j-1}, x_j)x_j + f(x_{j+1}, x_{j+2})x_{j+1} - f(x_j, x_{j+1})(x_j + x_{j+1})]\}, \end{aligned} \tag{6}$$

$$\begin{aligned} R_{3j}(x) &= y_j + f(x_j, x_{j+1})(x - x_j) \\ &+ \frac{2[y_j' - f(x_j, x_{j+1})]^2(x - x_j)(x_{j+1} - x)}{2[y_j' - f(x_j, x_{j+1})](x_{j+1} - x) + y_j''(x_j - x_{j+1})(x - x_j)}. \end{aligned} \tag{7}$$

Denote by $R_i(x)$ $(i = 1, 2, 3)$ the rational spline functions:

$$R_i(x) = \{R(x) \in (2,1)^1 |\ R(x)|_{[x_j, x_{j+1}]} = R_{ij}(x),\ j = 0, \cdots, n-1\},\ i = 1, 2;$$

$$R_3(x) = \{R(x) \in (2,1)^2 |\ R(x)|_{[x_j, x_{j+1}]} = R_{3j}(x),\ j = 0, \cdots, n-1\}.$$

It is not hard to prove the following lemmas:

[Lemma 1] $R_1(x)$ has the singular point in the interval $[x_j, x_{j+1}]$,if and only if

$$sign\{[y_j' - f(x_j, x_{j+1})] \cdot [y_{j+1}' - f(x_j, x_{j+1})]\} > 0. \tag{8}$$

[Lemma 2] $R_2(x)$ has the singular point in the interval $[x_j, x_{j+1}]$, if and only if

$$sign\{[f(x_{j-1}, x_j) - f(x_j, x_{j+1})] \cdot [f(x_{j+1}, x_{j+2}) - f(x_j, x_{j+1})]\} > 0. \tag{9}$$

In fact, by the formula of $R_1(x)$ on $[x_j, x_{j+1}]$ given in (5), the singularity of $R_1(x)$ on $[x_j, x_{j+1}]$ can appear only at point

$$x^* = \frac{[y_j' - f(x_j, x_{j+1})]x_j + [y_{j+1}' - f(x_j, x_{j+1})]x_{j+1}}{[y_j' - f(x_j, x_{j+1})] + [y_{j+1}' - f(x_j, x_{j+1})]} := \lambda x_j + (1-\lambda)x_{j+1},$$

where $\lambda = [y_j' - f(x_j, x_{j+1})]/[(y_j' - f(x_j, x_{j+1})) + (y_{j+1}' - f(x_j, x_{j+1}))]$. So,it is easy to see that $x^* \in (x_j, x_{j+1})$ if and only if

$$sign[(y_j' - f(x_j, x_{j+1})) \cdot (y_{j+1}' - f(x_j, x_{j+1}))] > 0.$$

By the similar argument, we can prove Lemma 2.

It notes that if $y_j' - f(x_j, x_{j+1})$ or $y_{j+1}' - f(x_j, x_{j+1}) = 0$, then $R_1(x)$ will be a linear function in the interval $[x_j, x_{j+1}]$, so it should has no singular point.

By using the above Lemmas, we have

[Theorem 1] Let the interpolation function $y = f(x) \in C^2[a, b]$. If $R_i(x)$ $(i = 1, 2)$ exists the singular point in the interval $[x_j, x_{j+1}]$, then $y = f(x)$ has the inflection point in the open interval (x_j, x_{j+1}).

Proof Let $R_1(x)$ exist the singular point in $[x_j, x_{j+1}]$. By Lemma 1,without loss the generality, suppose that the following inequalities hold

$$y_j' - f(x_j, x_{j+1}) > 0, \qquad y_{j+1}' - f(x_j, x_{j+1}) > 0. \tag{10}$$

It follows Lagrange's mean value theorem, that there exists $\xi \in (x_j, x_{j+1})$, such that

$$f'(\xi) = f(x_j, x_{j+1}). \tag{11}$$

By (10) and (11), there exist η and ζ of satisfying

$$f'(x_j) - f(x_j, x_{j+1}) = f''(\eta)(x_j - \xi), \qquad f'(x_{j+1}) - f(x_j, x_{j+1}) = f''(\zeta)(x_{j+1} - \xi)$$

respectively, where $x_j < \eta < \xi < \zeta < x_{j+1}$. Hence

$$f''(\eta) \cdot f''(\zeta) < 0,$$

and there exists at least one inflection point of $f(x)$ in (η, ζ). This completes the proof of this theorem for $R_1(x)$.

Similarly we can prove the theorem for the case of $R_2(x)$.

[Theorem 2] Let the interpolation function $y = f(x) \in C^2[a,b]$. If $R_3(x)$ exists the singular point in the interval $[x_j, x_{j+1}]$, then the original interpolation function $y = f(x)$ has the inflection point in the open interval (x_j, x_{j+1}).

In fact , because of the singular point of $R_3(x)$ may be only appearing at

$$\bar{x} = \frac{y_j''(x_{j+1} - x_j)x_j + 2(y_j' - f(x_j, x_{j+1}))x_{j+1}}{y_j''(x_{j+1} - x_j) + 2(y_j' - f(x_j, x_{j+1}))}.$$

By the same argument shown in the proof of Lemma 1, we have

$$sign\{[y_j''(x_{j+1} - x_j)] \cdot [y_j' - f(x_j, x_{j+1})]\} > 0,$$

provided that $R_3(x)$ exists the singular point in $[x_j, x_{j+1}]$.

It follows Lagrange's mean value theorem, that there exists $\xi \in (x_j, x_{j+1})$, such that

$$f'(\xi) = f(x_j, x_{j+1}).$$

By Lagrange's mean value theorem once again, there is a point $\eta \in (x_j, \xi)$, such that

$$f'(x_j) - f'(\xi) = f''(\eta)(x_j - \xi).$$

So

$$sign\{f''(x_j) \cdot f''(\eta)\} < 0.$$

Hence, there exists at least one inflection point of $f(x)$ in (x_j, η).

This complete the proof of this theorem.

In addition, we may prove that although the interpolation function $f(x)$ has the inflection points in $[a,b]$, however, provided that we are taking all inflection points as the knots of the rational spline function, then the singularity can be avoided to appear. For example, let x_j be an inflection point of $f(x)$, x_{j+1} be not, and there is no another inflection point between x_j and x_{j+1}. Then the first derivative of $f(x)$ will be monotone in the interval $[x_j, x_{j+1}]$. So the inequality (8) will be not satisfied. By Lemma 1, hence, there is no singular point in (x_j, x_{j+1}).

Therefore, we have

[Theorem 3] For any given interpolation function $f(x) \in C^2[a,b]$, we can construct a partition T of the interval$[a,b]$, such that the rational spline function $R_1(x)$, $R_2(x)$, and $R_3(x)$ based on the partition T have no singularity.

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References

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