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An Estimate on the Rate of Convergence of Viscosity Solutions for the Singular Perturbation Problems

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§1. Introduction

In this note we shall present a result on the rate of convergence of solutions for the singular perturbations of gradient obstacle problems. For any $\varepsilon > 0$, we consider the following nonlinear second-order elliptic partial differential equation (PDE);

$$(1.1)_\varepsilon \qquad \begin{cases} \max\{-\varepsilon^2\Delta u_\varepsilon + u_\varepsilon - f, |Du_\varepsilon| - g\} = 0 & \text{in} \quad \Omega, \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and f, g are nonnegative functions defined on $\overline{\Omega}$. This equation arises in some kind of stochastic control problem (cf. N. V. Krylov [9]). Our main purpose here is to get the optimal rate of convergence of solutions u_ε of $(1.1)_\varepsilon$ to the solution of u_0 of the first order PDE;

$$(1.1)_0 \qquad \begin{cases} \max\{u_0 - f, |Du_0| - g\} = 0 & \text{in} \quad \Omega, \\ u_0 = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

As to the equation $(1.1)_\varepsilon$, many authors discussed the existence and uniqueness of solutions. (See L. C. Evans [1], H. Ishii - S. Koike [4] and the second author [13].)

On the other hand, the estimate on the singular perturbation problems depend on complicated PDE or probabilistic techniques (e.g., S. R. S. Varadhan [12], and M. I. Freidlin - A. D. Wentzel [3]). However, here we shall obtain the estimate of point-wise convergence by a method easier than those. The method is an application of the comparison principle for viscosity solutions. (See H. Ishii - S. Koike [5].) Using the same method, S. Koike [8] has obtained the rate of convergence of solutions in singular perturbation problems. His result includes the singular perturbations of the obstacle problems, which are imposed to the unknown function itself.

Finally we give the definition of viscosity solution of general fully nonlinear second order elliptic PDEs. Consider

$$(1.2) \qquad F(x, u(x), Du(x), D^2u(x)) = 0 \qquad \text{in} \quad \Omega,$$

where F is a continuous function on $\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N$ ($\mathbb{S}^N$ denotes the set of all $N \times N$ real symmetric matrices) satisfying the following ellipticity condition;

$$F(x, r, p, A + B) \leqq F(x, r, p, A) \qquad \text{for all } x \in \Omega,\ r \in \mathbb{R},$$
$$p \in \mathbb{R}^N,\ A, B \in \mathbb{S}^N \text{ and } B \geqq O.$$

For the function u defined on $\overline{\Omega}$, let u^* (resp. u_*) be the upper (resp. lower) semi-continuous envelope of u on $\overline{\Omega}$;

$$u^*(x) = \lim_{r \to 0} \sup\{u(y) \mid |y - x| < r,\ y \in \overline{\Omega}\},$$
$$u_*(x) = \lim_{r \to 0} \inf\{u(y) \mid |y - x| < r,\ y \in \overline{\Omega}\}.$$

Definition. *Let u be a function defined on $\overline{\Omega}$.*

(1) *u is a viscosity subsolution of* (1.2) *provided $u^*(x) < +\infty$ in Ω and for any $\varphi \in C^2(\Omega)$, if $u^* - \varphi$ attains a local maximum at $x_0 \in \Omega$, then*

$$F(x_0, u^*(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leqq 0.$$

(2) *u is a viscosity supersolution of* (1.2) *provided $u_*(x) > -\infty$ in Ω and for any $\varphi \in C^2(\Omega)$, if $u_* - \varphi$ attains a local minimum at $x_0 \in \Omega$, then*

$$F(x_0, u_*(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geqq 0.$$

(3) *u is a viscosity solution of* (1.2) *provided u is a viscosity subsolution and a supersolution of* (1.2)

Remark. (i) In the case of first order PDEs, we can replace $C^2(\Omega)$ in (1) or (2) with $C^1(\Omega)$.

(ii) For the details, see H. Ishii - P. L. Lions [6].

§2. Preliminaries

In this section we shall state our assumptions and shall show the existence and uniqueness of viscosity solutions of $(1.1)_\varepsilon$ and $(1.1)_0$ satisfying the Dirichlet boundary condition. We make the following assumptions.

(A.1) $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$.

(A.2) $f \in W^{1,\infty}(\overline{\Omega})$ and $f \geqq 0$ on $\overline{\Omega}$.

(A.3) $g \in W^{1,\infty}(\overline{\Omega})$ and $g \geqq \theta$ on $\overline{\Omega}$ for some $\theta > 0$.

We denotes by K_f and K_g the Lipschitz constants of f and g, respectively.

Concerning the existence and uniqueness of viscosity solutions of $(1.1)_\varepsilon$ and $(1.1)_0$ satisfying the Dirichlet boundary condition, we have the following Theorem.

Theorem 1. (1) *For each $\varepsilon > 0$, there exists a unique viscosity solution $u_\varepsilon \in W^{1,\infty}(\overline{\Omega})$ of $(1.1)_\varepsilon$ satisfying the Dirichlet boundary condition.*

(2) *There exists a unique viscosity solution $u_0 \in W^{1,\infty}(\overline{\Omega})$ of $(1.1)_0$ satisfying the Dirichlet boundary condition.*

PROOF: The uniqueness of viscosity solutions follows from the comparison principle due to H. Ishii - P. L. Lions [6].

Next we show the existence of solutions. We note that by (A.2) and (A.3),

$$w_1(x) = 0 \qquad \text{on} \quad \overline{\Omega} \tag{2.1}$$

is a viscosity subsolution of $(1.1)_\varepsilon$ and $(1.1)_0$. On the other hand, P. L. Lions [11] proved that

$$w_2(x) = \inf_{y \in \partial\Omega} L(x, y) \qquad \text{on} \quad \overline{\Omega}, \tag{2.2}$$

is a viscosity supersolution of $(1.1)_\varepsilon$ and $(1.1)_0$, where

$$L(x, y) = \inf_{\xi \in \mathcal{A}} \int_0^t g(\xi(s))ds,$$

$$\mathcal{A} = \left\{ \xi \in C[0,t] \;\middle|\; \xi(0) = x, \xi(t) = y \in \partial\Omega, \right.$$

$$\left. \xi(s) \in \overline{\Omega} \ \ (0 \leqq s \leqq t), \left|\frac{d\xi}{ds}\right| \leqq 1 \quad \text{a.e. } s \in [0,t] \right\}.$$

Thus by Perron's method there exist viscosity solutions u_ε, $u_0 \in C(\overline{\Omega})$ of $(1.1)_\varepsilon$, $(1.1)_0$ respectively satisfying the Dirichlet boundary condition and

$$0 \leqq u_\varepsilon,\ u_0 \leqq w_2 \qquad \text{on} \quad \overline{\Omega}. \tag{2.3}$$

Moreover the form of equations $(1.1)_\varepsilon$ and $(1.1)_0$ implies that u_ε and u_0 are viscosity subsolutions of $|Du| - g = 0$ in Ω. Hence it follows from M. G. Crandall - P. L. Lions [2] that u_ε and u_0 are Lipschitz continuous on $\overline{\Omega}$. Therefore we complete the proof. ∎

Remark. (i) In order to show the comparison principle, it is sufficient to assume f, $g \in C(\overline{\Omega})$.
(ii) Since g is a bounded constraint for the gradient of u_ε, the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ are equi-Lipschitz continuous on $\overline{\Omega}$. In what follows K denotes the Lipschitz constant of u_ε and u_0.

§3. Main result

This section is devoted to our main result.

Theorem 2. *We assume* (A.1)-(A.3). *Let u_ε, u_0 be viscosity solutions of* $(1.1)_\varepsilon$, $(1.1)_0$ *respectively satisfying the Dirichlet boundary condition. Then there exist $\varepsilon_0 > 0$ and $\mu > 0$ such that*

$$\|u_\varepsilon - u_0\| \leqq \mu\varepsilon \qquad \textit{for all } \varepsilon \in (0, \varepsilon_0),$$

where $\|\cdot\|$ denotes the supremum norm in $C(\overline{\Omega})$.

Before proving Theorem 2, we shall give an example. It shows that the above estimate is optimal.

Example. Let $\Omega = (-1, 1)$, $f(x) = 1 - |x|$, and $g \equiv 1$ on $\overline{\Omega}$. Then we have viscosity solutions u_ε, u_0 of $(1.1)_\varepsilon$, $(1.1)_0$ as follows;

$$u_\varepsilon(x) = \varepsilon\frac{\sinh((|x| - 1)/\varepsilon)}{\cosh(1/\varepsilon)} + 1 - |x|,$$
$$u_0(x) = 1 - |x|.$$

We note that $\tanh x < 1$ and $\tanh x \to 1 \quad (x \to +\infty)$. Thus we get the following estimate;

$$\|u_\varepsilon - u_0\| = |u_\varepsilon(0) - u_0(0)| = \varepsilon \tanh(1/\varepsilon) \leqq \varepsilon \qquad \text{for } 0 < \varepsilon < 1.$$

PROOF OF THEOREM 2: It is sufficient to prove the upper estimate $u_\varepsilon - u_0 \leqq \mu\varepsilon$ on $\overline{\Omega}$ because the lower estimate $-\mu\varepsilon \leqq u_\varepsilon - u_0$ on $\overline{\Omega}$ can be proved similarly. We take $\varepsilon_0 > 0$ such that

$$\varepsilon_0 = \frac{\theta}{3K_g K}$$

and for each $\varepsilon \in (0, \varepsilon_0)$, we define

$$\Phi_\varepsilon(x,y) = \rho u_\varepsilon(x) - u_0(y) - \frac{|x-y|^2}{\varepsilon} - \mu\varepsilon \qquad \text{on} \quad \overline{\Omega \times \Omega},$$

where $\rho = 1 - 3K_g K\varepsilon/2\theta$ and $\mu > 0$ is a constant to be determined later. Let $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega \times \Omega}$ be a maximum point of the function $\Phi_\varepsilon(x,y)$. Then $\Phi_\varepsilon(x_\varepsilon, x_\varepsilon) \leqq \Phi_\varepsilon(x_\varepsilon, y_\varepsilon)$ and we get

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} \leqq u_0(x_\varepsilon) - u_0(y_\varepsilon).$$

Since u_0 is Lipschitz continuous, we have

$$|x_\varepsilon - y_\varepsilon| \leqq K\varepsilon. \tag{3.2}$$

We consider the following three cases.

Case 1. $x_\varepsilon, y_\varepsilon \in \Omega$.

The function

$$x \to u_\varepsilon(x) - \frac{1}{\rho}\left\{u_0(y_\varepsilon) + \frac{|x - y_\varepsilon|^2}{\varepsilon} + \mu\varepsilon\right\}$$

takes the maximum at x_ε. Similarly, the function

$$y \to u_0(y) - \left\{\rho u_\varepsilon(x_\varepsilon) - \frac{|x_\varepsilon - y|^2}{\varepsilon} - \mu\varepsilon\right\}$$

takes the minimum at y_ε. Hence regarding u_ε as a viscosity subsolution of $(1.1)_\varepsilon$ and u_0 as a viscosity supersolution of $(1.1)_0$, we obtain two inequalities;

$$\max\left\{-\frac{2N}{\rho}\varepsilon + u_\varepsilon(x_\varepsilon) - f(x_\varepsilon), \frac{2|x_\varepsilon - y_\varepsilon|}{\rho\varepsilon} - g(x_\varepsilon)\right\} \leqq 0, \tag{3.3}$$

$$\max\left\{u_0(y_\varepsilon) - f(y_\varepsilon), \frac{2|x_\varepsilon - y_\varepsilon|}{\varepsilon} - g(y_\varepsilon)\right\} \geqq 0. \tag{3.4}$$

We claim that $2|x_\varepsilon - y_\varepsilon|/\varepsilon - g(y_\varepsilon) < 0$ in (3.4). To prove the inequality by contradiction, suppose that $2|x_\varepsilon - y_\varepsilon|/\varepsilon - g(y_\varepsilon) \geqq 0$ in (3.4). Since $2|x_\varepsilon - y_\varepsilon|/\rho\varepsilon - g(x_\varepsilon) \leqq 0$ by (3.3), we get

$$g(y_\varepsilon) \leqq \frac{2|x_\varepsilon - y_\varepsilon|}{\varepsilon} \leqq \rho g(x_\varepsilon).$$

Thus (A.3) and (3.2) imply that

$$(1-\rho)\theta \leqq (1-\rho)g(y_\varepsilon) \leqq \rho(g(x_\varepsilon) - g(y_\varepsilon)) \leqq K_g|x_\varepsilon - y_\varepsilon| \leqq K_g K\varepsilon.$$

Hence we have $3/2 \leqq 1$, which is a contradiction. Therefore we obtain the claim.

Thus we get from (3.4)

$$u_0(y_\varepsilon) - f(y_\varepsilon) \geqq 0. \tag{3.5}$$

Note that (3.3) implies

$$-\frac{2N}{\rho}\varepsilon + u_\varepsilon(x_\varepsilon) - f(x_\varepsilon) \leqq 0. \tag{3.6}$$

Subtracting (3.5) from (3.6) and using (3.1), (3.2) and (A.2), we have

$$\begin{aligned} u_\varepsilon(x_\varepsilon) - u_0(y_\varepsilon) &\leqq \frac{2N}{\rho}\varepsilon + f(x_\varepsilon) - f(y_\varepsilon) \\ &\leqq C\varepsilon + K_f|x_\varepsilon - y_\varepsilon| \\ &\leqq C\varepsilon. \end{aligned}$$

Here and hereafter C denotes various constants depending only on known constants. Hence we obtain

$$\begin{aligned} \rho u_\varepsilon(x) - u_0(x) - \mu\varepsilon = \Phi_\varepsilon(x,x) &\leqq \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \\ &\leqq u_\varepsilon(x_\varepsilon) - u_0(y_\varepsilon) - \mu\varepsilon \\ &\leqq (C-\mu)\varepsilon. \end{aligned}$$

Now we choose $\mu > 0$ large enough to get $\rho u_\varepsilon(x) - u_0(x) \leqq \mu\varepsilon$. Therefore

$$u_\varepsilon(x) - u_0(x) \leqq \left(\mu + \frac{3K_gK}{2\theta}u_\varepsilon(x)\right)\varepsilon$$
$$\leqq (\mu + C)\varepsilon.$$

Replacing μ with $\mu + C$, we have the upper estimate.

Case 2. $x_\varepsilon \in \partial\Omega$.

Since the Dirichlet boundary conditon of $(1.1)_\varepsilon$ and (2.3) imply

$$\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) = -u_0(y_\varepsilon) - \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} - \mu\varepsilon \leqq 0$$

for any $\mu > 0$, we can argue the remainder similar to *Case 1*.

Case 3. $y_\varepsilon \in \partial\Omega$.

By the Dirichlet boundary condition of $(1.1)_\varepsilon$ and $(1.1)_0$ and the equi-Lipschitz continuity of $\{u_\varepsilon\}_{\varepsilon>0}$, we obtain

$$\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \rho u_\varepsilon(x_\varepsilon) - \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} - \mu\varepsilon$$
$$\leqq u_\varepsilon(x_\varepsilon) - u_\varepsilon(y_\varepsilon) - \mu\varepsilon$$
$$\leqq (K^2 - \mu)\varepsilon.$$

Thus we get $\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \leqq 0$ for $\mu \geqq K^2$. The remainder is also proved similarly to *Case 1*.

From *Case 1* to *Case 3*, if we choose $\mu > 0$ sufficiently large, then we have the upper estimate;

$$u_\varepsilon(x) - u_0(x) \leqq \mu\varepsilon \qquad \text{for all } x \in \overline{\Omega}.$$

Replacing u_ε and u_0 with each other in the above argument, we obtain the lower estimate;

$$-\mu\varepsilon \leqq u_\varepsilon(x) - u_0(x) \qquad \text{for all } x \in \overline{\Omega}.$$

Hence we complete the proof. ∎

Final Remark. Under some reasonable assumptions, we can extend Theorem 2 to the following equations.

(1) *Hamilton-Jacobi-Bellman equation with gradient constraint;*

$$\begin{cases} \max\{L_\varepsilon^1 u_\varepsilon - f^1, \cdots, L_\varepsilon^m u_\varepsilon - f^m, |Du_\varepsilon| - g\} = 0 & \text{in} \quad \Omega, \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where L_ε^p $(p = 1, \cdots, m)$ are linear second order elliptic operators defined in $\Omega \subset \mathbb{R}^N$;

$$L_\varepsilon^p u = -\varepsilon^2 a_{ij}^p u_{x_i x_j} + \varepsilon b_i^p u_{x_i} + c^p u,$$

and f^p, g are nonnegative functions on $\overline{\Omega}$. The corresponding first order PDE is as follows;

$$\begin{cases} \max\{c^1 u_0 - f^1, \cdots, c^m u_0 - f^m, |Du_0| - g\} = 0 & \text{in} \quad \Omega, \\ u_0 = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

(2) *Second order elliptic PDE with gradient constraint whose principal part is a fully nonlinear operator;*

$$\begin{cases} \max\{F(x, u_\varepsilon, \varepsilon Du_\varepsilon, \varepsilon^2 D^2 u_\varepsilon), |Du_\varepsilon| - g\} = 0 & \text{in} \quad \Omega, \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and the first order PDE;

$$\begin{cases} \max\{F(x, u_0, 0, O), |Du_0| - g\} = 0 & \text{in} \quad \Omega, \\ u_0 = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $F(x, r, p, A)$ is continuous on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N$ and nonincreasing with respect to the variable $A \in \mathbb{S}^N$.

See the authors [7] for the details.

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