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射影空間内の点の配置とtheta関数

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In this note, we announce a structure theorem on the graded ring of modular forms on the bounded symmetric domain of type $I_{2,2}$ with respect to the principal congruence subgroup of level (1+i). Relations between the period map of certain K3 surfaces, the hypergeometric functions and the present theorem have been studied in [MSY2].

We first explain the classical model, which prepares the understanding of our case. Theta functions $\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau)$ on the upper half plane $\mathbb{H} = \{\tau\in\mathbb{C} \mid \operatorname{Im}\tau > 0\}$ are defined by

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau) := \sum_{n\in\mathbb{Z}} e[(n+\tfrac{1}{2}a)^2\tau + bn],$$

where $\tau\in\mathbb{H}$, $a,b\in\mathbb{Z}$ and $e[x] = \exp(\pi i x)$. These functions have the following properties:

(i) $\quad \vartheta\begin{bmatrix}a+r\\b+s\end{bmatrix}(\tau) = \exp(\tfrac{1}{2}\pi i b\cdot r)\,\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau)$, where $r,s\in 2\mathbb{Z}$,

(ii) if ${}^t a\cdot b \notin 2\mathbb{Z}$, then $\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau)$ vanishes.

By the properties (i) and (ii), it is enough to consider only three theta functions $\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau)$, where $a,b\in\{0,1\}$ and $a\cdot b = 0$. The group $SL(2,\mathbb{R})$ acts on $\mathbb{H}$ by

$$g\cdot\tau = (A\tau+B)(C\tau+D)^{-1},\ \tau\in\mathbb{H},\ g = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\in SL(2,\mathbb{R}).$$

Let $\Gamma(2)$ be the congruence subgroup of level 2:

$$\Gamma(2) = \{g\in SL(2,\mathbb{Z}) \mid g \equiv I_2 \mod 2\}.$$

Definition. A holomorphic function f on $\mathbb{H}$ is called a modular form of weight k relative to $\Gamma(2)$, if the following condition is satisfied:

$$f(g\cdot\tau) = \{\det(C\tau+D)\}^k f(W), \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma(2).$$

Let $\mathrm{Mod}_k(2)$ denote the vector space spaned by modular forms of weight k relative to $\Gamma(2)$ and let

$$\mathrm{Mod}(2) := \bigoplus_{k\geq 0} \mathrm{Mod}_k(2)$$

be the graded ring of modular forms relative to $\Gamma(2)$. It is well known that

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\tau)^4 \in \mathrm{Mod}_2(2),$$

where $a, b \in \{0,1\}$ and $a\cdot b = 0$, and that these functions satisfy the following relation:

$$\vartheta\begin{bmatrix} 0 \\ 1 \end{bmatrix}(\tau)^4 - \vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau)^4 + \vartheta\begin{bmatrix} 1 \\ 0 \end{bmatrix}(\tau)^4 = 0.$$

Let $\overline{\Gamma(2)\backslash\mathbb{H}}$ be the compactification of the quotient space $\Gamma(2)\backslash\mathbb{H}$. We define a holomorphic map ϕ of $\overline{\Gamma(2)\backslash\mathbb{H}}$ into the complex projective plane $\mathbb{P}^2$ by

$$\phi : \tau \mapsto [\vartheta\begin{bmatrix} 0 \\ 1 \end{bmatrix}(\tau)^4, \ \vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau)^4, \ \vartheta\begin{bmatrix} 1 \\ 0 \end{bmatrix}(\tau)^4].$$

By the above relation, the image of ϕ is included in a line in $\mathbb{P}^2$, which will be denoted by X. Moreover, the map ϕ gives an isomorphism between $\overline{\Gamma(2)\backslash\mathbb{H}}$ and X ($\simeq \mathbb{P}^1$). This fact implies that the graded ring $\mathrm{Mod}(2)$ is generated by $\vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau)^4$ and $\vartheta\begin{bmatrix} 0 \\ 1 \end{bmatrix}(\tau)^4$.

Next we explain our case. A classical bounded symmetric domain of type $I_{2,2}$ is defined by

$$D := \{\ W = (w_{jk})_{1\leq j,k\leq 2} \mid \frac{W - W^*}{2i} > 0\ \}, \text{ where } W^* = {}^t\overline{W}.$$

The group Aut(D) of automorphisms of D is generated by U(2,2) and T, where

$$T(W) = {}^tW,$$

$$U(2,2) = \{g \in GL(4,\mathbb{C}) \mid g^* J g = J,\ J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}\},$$

which acts on D by

$$(g\cdot T^j)\cdot W = \{A(T^j\cdot W)+B\}\{C(T^j\cdot W)+D\}^{-1},$$

where $W \in D$, $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(2,2)$, $j \in \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$; hence we have

$$Aut(D) \simeq U(2,2) \rtimes \langle T\rangle,\quad \langle T\rangle = \{id, T\}.$$

We define theta functions on D as follows:

$$\Theta\begin{bmatrix} a \\ b \end{bmatrix}(W) := \sum_{n\in\mathbb{Z}[i]^2} e[(n + \tfrac{1}{1+i}a)^* W (n + \tfrac{1}{1+i}a) + 2\mathrm{Re}((\tfrac{1}{1+i}b)^* n)],$$

where $W \in D$ and $a,b \in \mathbb{Z}[i]^2$. These functions have the following properties:

(i) $\Theta\begin{bmatrix} a \\ b \end{bmatrix}(W) = \Theta\begin{bmatrix} a \\ b \end{bmatrix}({}^tW)$,

(ii) $\Theta\begin{bmatrix} \delta a \\ \varepsilon b \end{bmatrix}(W) = \Theta\begin{bmatrix} a \\ b \end{bmatrix}(W)$, where δ and ε are units of $\mathbb{Z}[i]$,

(iii) $\Theta\begin{bmatrix} a+r \\ b+s \end{bmatrix}(W) = \exp(\pi i\ \mathrm{Re}\,{}^tb\cdot r)\ \Theta\begin{bmatrix} a \\ b \end{bmatrix}(W)$, where $r,s \in (1+i)\mathbb{Z}[i]^2$,

(iv) if ${}^ta\cdot b \notin (1+i)\mathbb{Z}[i]$, then $\Theta\begin{bmatrix} a \\ b \end{bmatrix}(W)$ vanishes.

By the properties (ii),(iii) and (iv), it is enough to consider only ten theta functions $\Theta\begin{bmatrix} a \\ b \end{bmatrix}(W)$, where $a,b \in \{0,1\}^2$ and ${}^ta\cdot b = 0$. Let $\Gamma(1+i)$ be the congruence subgroup

$$\Gamma(1+i) := \{g \in GL(4,\mathbb{Z}[i]) \mid g^* J g = J,\ g \equiv I_4 \mod (1+i)\},$$

of level (1+i), and let

$$\Gamma_T(1+i) := \Gamma(1+i) \rtimes \langle T\rangle \subset Aut(D).$$

<u>Definition</u>. A holomorphic function f on D is called a <u>hermitian</u>

<u>modular form of weight</u> 2k <u>relative to</u> $\Gamma_T(1+i)$ (with the character $\det: \Gamma(1+i) \to$ the group of units of $\mathbb{Z}[i]$), if the following conditions are satisfied:

(i) $f(T\cdot W) = f(W)$,

(ii) $f(g\cdot W) = (\det(g))^k \, (\det(CW+D))^{2k} \, f(W)$, $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma(1+i)$.

Let $\mathrm{Mod}_{2k}(1+i)$ denote the vector space spaned by hermitian modular forms of weight 2k relative to $\Gamma_T(1+i)$ and let

$$\mathrm{Mod}(1+i) := \bigoplus_{k\geq 0} \mathrm{Mod}_{2k}(1+i)$$

be the graded ring of hermitian modular forms relative to $\Gamma_T(1+i)$.

<u>Proposition 1</u>. <u>The ten functions</u> $\Theta\begin{bmatrix} a \\ b \end{bmatrix}(W)^2$ <u>$(a,b \in \{0,1\}^2,\ {}^t a\cdot b = 0)$ are hermitian modular forms of weight 2 relative to</u> $\Gamma_T(1+i)$.

Proposition 1 is essentially proved in [Fr].

<u>Proposition 2</u>. <u>Theta functions</u> $\Theta\begin{bmatrix} a \\ b \end{bmatrix}(W)$ <u>$(a,b \in \{0,1\}^2,\ {}^t a\cdot b = 0)$ satisfy the following relations</u>:

$$\sum_{{}^t a\cdot b \in 2\mathbb{Z}} \Theta\begin{bmatrix} a \\ b \end{bmatrix}(W)^2 \, e[{}^t ca + {}^t db] = 0, \quad c,d \in \{0,1\}^2,\ {}^t c\cdot d = 1.$$

<u>Remark 3</u>. The proposition gives six linear relations between the theta functions $\Theta\begin{bmatrix} a \\ b \end{bmatrix}(W)^2$; five relations among the six are linearly independent.

Let $\overline{\Gamma_T(1+i)\backslash D}$ be the Satake compactification of the quotient space $\Gamma_T(1+i)\backslash D$. We define a holomorphic map $\Phi: \overline{\Gamma_T(1+i)\backslash D} \to \mathbb{P}^9$ by

$$\Phi: W \mapsto [\cdots, \Theta\begin{bmatrix} a \\ b \end{bmatrix}(W)^2, \cdots].$$

By Remark 3, it is clear that the image of Φ is included in a 4-dimensional linear subspace of $\mathbb{P}^9$, which will be denoted by Y. Now we state the main result of the present paper:

Theorem 4. The map Φ gives an isomorphism between $\overline{\Gamma_T(1+i)\backslash D}$ and $Y\ (\simeq \mathbb{P}^4)$.

Corollary 5. Any five linearly independent theta functions $\Theta\begin{bmatrix}a\\b\end{bmatrix}(W)^2$ are free generators of the graded ring Mod(1+i).

Remark 6. As the isomorphism $\phi: \overline{\Gamma(2)\backslash \mathbb{H}} \to X\ (\simeq \mathbb{P}^1)$ connects the analytic moduli and the algebraic moduli of the family of ellipitic curves, the isomorphism $\Phi: \overline{\Gamma_T(1+i)\backslash D} \to Y\ (\simeq \mathbb{P}^4)$ connects the analytic moduli and the algebraic moduli of a 4-dimensional family of K3 surfaces, which are double cover of $\mathbb{P}^2$ branching along 6 lines. For more details, see [MSY2] and [Ma2].

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