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Algebraic versus rigid cohomology with logarithmic coefficients: the 1-dimensional example.

by

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§1 Notation.

$(K, |-|)$= a complete, algebraically closed valued field extension of $(\mathbf{Q}_p, |-|)$, for some prime p ; $|p| = p^{-1}$.

$\mathcal{V}$= the ring of integers of $(K, |-|)$.

$\mathcal{M}$= the maximal ideal of $\mathcal{V}$.

$k = \mathcal{V}/\mathcal{M}$, the residue field of K.

For a $\mathcal{V}$-scheme T of finite presentation, we put:

$T_\circ = T \times_\mathcal{V} k$, the special fiber of T

$T_K = T \times_\mathcal{V} K$, the generic fiber of T.

$\hat{T}$: the formal completion of T along $T_\circ$.

T_K^{an}= the rigid analytic space associated to the K-scheme T_K ([BGR]).

For a p-adic formal $\mathcal{V}$-scheme $\mathcal{T}$ of finite presentation, we put:

$\mathcal{T}_\circ = \mathcal{T} \times_\mathcal{V} k$, the special fiber of $\mathcal{T}$.

$\mathcal{T}_K = \mathcal{T} \times_\mathcal{V} K$ = the generic fiber of $\mathcal{T}$ in the sense of Raynaud and Berthelot ([Ra], [Ber]): it's a rigid analytic space.

For a separated $\mathcal{V}$-scheme of finite presentation T, using the previous notation, we get an open immersion of rigid analytic spaces:

$$\hat{T}_K \longrightarrow T_K^{an} ,$$

which is an isomorphism when T is proper over $\mathcal{V}$ ([Be]).

The following definition will play an important role in the sequel.

Definition 1.1 For $\gamma \in K$ the *type* $\rho(\gamma)$ of γ is the radius of convergence of the series:

$$g_\gamma(x) = \sum_{\gamma \neq i=0}^{\infty} \frac{x^i}{\gamma - i}.$$

Notice that $\rho(\gamma) \in [0,1]$ and $\rho(\gamma) = \rho(\gamma + n)$, $\forall n \in \mathbf{Z}$. We say that γ is p-adically non-Liouville if $\rho(\gamma) = \rho(-\gamma) = 1$.

Remark. Algebraic numbers are p-adically non-Liouville.

§2 Main Result.

We consider:
Y = a proper smooth, connected $\mathcal{V}$-scheme.
Z = a divisor in Y with normal crossing relative to $\mathcal{V}$:

$$Z = \bigcup_{i=1}^{r} Z^{(i)}$$

where $Z^{(i)}$ is a closed $\mathcal{V}$-subscheme of Y, smooth, connected of codimension 1.

X = the open $\mathcal{V}$-subscheme of Y, complementary to Z in Y.

The previous hypotheses mean that there exists a finite covering $\mathcal{U}$ of Y by affine open subsets U such that:

i) U is étale over $\mathbf{A}^m_{\mathcal{V}}$ via "coordinates" $(x_1, \ldots, x_m)$.

ii) The ideal of $Z_U = Z \times_Y U = Z_{|U}$ in $\mathcal{O}(U)$ is generated by $x_1 \ldots x_\nu = 0$ where $\nu = \nu(U)$.

We also consider:
$\mathcal{E}_{\mathcal{V}}$ = a locally free finite $\mathcal{O}_Y$-module.
∇ = an integrable Y_K/K connection on $\mathcal{E} = \mathcal{E}_{\mathcal{V}} \otimes K$ with logarithmic singularities along Z_K.
So, ∇ is a morphism of abelian sheaves:

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_{Y_K/K} < Z_K >$$

satisfying Leibnitz's rule and the usual integrability condition. We recall that using the previous notation we have, on $U \in \mathcal{U}$ as before:

$$\Omega^1_{Y_K/K} < Z_K >_{|U_K} = \sum_{i=1}^{\nu} \mathcal{O}_{U_K} \frac{dx_i}{x_i} + \sum_{j=\nu+1}^{m} \mathcal{O}_{U_K} dx_j.$$

The hypercohomology of the de Rham complex of $(\mathcal{E}, \nabla)_{|X_K}$ i.e. of the complex:

$$\mathcal{DR}(X_K/K, (\mathcal{E}, \nabla)) \; = \; 0 \rightarrow \mathcal{E}_{|X_K} \longrightarrow \mathcal{E}_{|X_K} \otimes \Omega^1_{X_K/K} \longrightarrow \ldots$$

is, by definition, the algebraic cohomology of X_K with coefficients in $(\mathcal{E}, \nabla)$, denoted by:

$$\mathbf{H}^\bullet(X_K/K, (\mathcal{E}, \nabla)).$$

¿From the morphism of ringed sites:

$$\beta : X_K^{an} \longrightarrow X_K.$$

one may deduce from $(\mathcal{E}, \nabla)$ a X_K^{an}/K-connection $(\mathcal{E}^{an}, \nabla^{an})$. The hypercohomology of its de Rham complex

$$\mathcal{DR}(X_K^{an}/K, (\mathcal{E}^{an}, \nabla^{an})$$

is, by definition, the analytic cohomology (in the rigid analytic sense) of X_K^{an} with coefficients in $(\mathcal{E}^{an}, \nabla^{an})$, denoted by:

$$\mathbf{H}^\bullet(X_K^{an}/K, (\mathcal{E}^{an}, \nabla)^{an}).$$

The morphism of ringed sites, β, induces a natural morphism of complexe s of sheaves :

$$\beta^{-1}\mathcal{DR}(X_K/K, (\mathcal{E}, \nabla)) \longrightarrow \mathcal{DR}(X_K^{an}/K, (\mathcal{E}^{an}, \nabla^{an}))$$

which gives a morphism in hypercohomology:

$$\overline{\beta}: \quad \mathbf{H}^\bullet(X_K/K, (\mathcal{E}, \nabla)) \longrightarrow \mathbf{H}^\bullet(X_K^{an}/K, (\mathcal{E}^{an}, \nabla)^{an}).$$

Consider now the assumption:

$(\mathbf{NL})_{\mathbf{G}}$ The additive subgroup $\Lambda$of K generated by the exponents of monodromy of $(\mathcal{E}, \nabla)$ around the branches of Z_K consists of p-adically non-Liouville numbers.

In this setting we proved years ago ([Ba2]) a result of GAGA type :

Theorem 2.1. *Under the assumption* $(\mathbf{NL})_{\mathbf{G}}$ *the morphism* $\overline{\beta}$ *is an isomorphism.*

Other results in the same direction may be found in [Bal],[Ct].

We now come to our main statement. Let

$$j_\circ : X_\circ \longrightarrow Y_\circ,$$

$$j : X_K^{an} \longrightarrow Y_K^{an}$$

denote the corresponding open immersions. We notice that X_K^{an} is a strict neighborhood of the tube $]X_\circ[$ of $X_\circ$ in $\hat{Y}_K = Y_K^{an}$ [Ber]. Using the theory of Berthelot we may consider the $j_\circ^\dagger$-completion of the previous coefficients. We recall the definition. For $\lambda \in (0,1)$ sufficiently close to 1, we define

$$\mathbf{V}_\lambda = \hat{Y}_K \setminus \bigcup_{i=1}^{r} [Z_\circ^{(i)}]_\lambda$$

where $[Z_\circ^{(i)}]_\lambda$ denotes the closed tube of radius λ of $Z_\circ^{(i)}$ in $\hat{Y}_K$.
We get open immersions:

$$\begin{array}{ccc} \mathbf{V}_\lambda & \xrightarrow{j_\lambda} & \hat{Y}_K \\ \downarrow & \nearrow & \\ X_K^{an} & & \end{array}$$

We denote by $(\mathcal{E}_\lambda, \nabla_\lambda)$ the connection induced by $(\mathcal{E}^{an}, \nabla^{an})$ on $\mathbf{V}_\lambda$, and by $\mathcal{DR}(\mathbf{V}_\lambda/K, (\mathcal{E}_\lambda, \nabla_\lambda)) = \mathcal{DR}_\lambda$ its de Rham complex (i.e. the restriction of $\mathcal{DR}(X_K^{an}/K, (\mathcal{E}^{an}, \nabla^{an}))$ to $\mathbf{V}_\lambda$.

We then obtain a connection $(\mathcal{E}^\dagger, \nabla^\dagger)$ on

$$\mathcal{E}^\dagger = j_\circ^\dagger \mathcal{E} =_{def} \varinjlim_{\lambda \to 1^-} j_{\lambda *}\mathcal{E}_\lambda$$

whose de Rham complex is:

$$\mathcal{DR}(\hat{Y}_K/K, (\mathcal{E}^\dagger, \nabla^\dagger)) = \varinjlim_{\lambda \to 1^-} j_{\lambda *}\mathcal{DR}_\lambda.$$

We introduce the hypothesis:

$(\mathbf{SC})_{\mathbf{G}}$ The connection $(\mathcal{E}^\dagger, \nabla^\dagger)$ is overconvergent along $Z_\circ$.

As in [Ber], we define the rigid cohomology of $X_\circ$ with coefficients in $(\mathcal{E}^\dagger, \nabla^\dagger)$ as:

$$H^\bullet_{rig}(X_\circ/K, (\mathcal{E}^\dagger, \nabla^\dagger)) = \mathbf{H}^\bullet(\hat{Y}_K, \mathcal{DR}(\hat{Y}_K/K, (\mathcal{E}^\dagger, \nabla^\dagger)).$$

We also notice that, since $\hat{Y}_K$ is quasi-compact and separated, the cohomology on $\hat{Y}_K$ commutes with the direct limits ([Ber], [SGAIV]). Hence :

$$(2.2) \qquad H^\bullet_{rig}(X_\circ/K, (\mathcal{E}^\dagger, \nabla^\dagger)) = \mathbf{H}^\bullet(\hat{Y}_K, \varinjlim_{\lambda} j_{\lambda *}\mathcal{DR}_\lambda) = \varinjlim_{\lambda} \mathbf{H}^\bullet(\hat{Y}_K, j_{\lambda *}\mathcal{DR}_\lambda).$$

We then have a natural morphism of complexes of sheaves on $\hat{Y}_K = Y_K^{an}$:

$$\begin{array}{ccc} j_{\lambda *}\mathcal{DR}_\lambda & \longrightarrow & \mathcal{DR}(\hat{Y}_K/K, (\mathcal{E}^\dagger, \nabla^\dagger)) \\ \downarrow & \nearrow & \\ j_*\mathcal{DR}(X_K^{an}/K, (\mathcal{E}^{an}, \nabla^{an})) & & \end{array}$$

and the induced morphism in hypercohomology:

$$\begin{array}{ccc} \mathbf{H}^\bullet(\mathbf{V}_\lambda/K, (\mathcal{E}_\lambda, \nabla_\lambda)) & \longrightarrow & H^\bullet_{rig}(X_\circ/K, (\mathcal{E}^\dagger, \nabla^\dagger)) \\ \downarrow \overline{\alpha}_\lambda & \nearrow & \\ \mathbf{H}^\bullet(X_K^{an}/K, (\mathcal{E}^{an}, \nabla^{an})) & & \end{array}$$

(this makes sense because $R^q j_* \mathcal{F} = 0$ (resp. $R^q j_{\lambda *}\mathcal{F} = 0$) for any coherent sheaf $\mathcal{F}$ on X_K^{an} (resp. $\mathbf{V}_\lambda$) and $q > 0$, since j_* (resp. $j_{\lambda *}$) is a quasi-Stein map [Ba1],[K]).

Our main result is the following:

Theorem 2.4. *Under the assumptions* $(\mathrm{NL})_{\mathbf{G}}$ *and* $(\mathrm{SC})_{\mathbf{G}}$ *the morphism* $\overline{\alpha}_\lambda$ *is an isomorphism for any* $\lambda \in (0,1)$ *for which* $\mathbf{V}_\lambda$ *is defined (i.e. for any* λ *sufficiently close to 1).*

From (2.2) and the identification :

$$\mathbf{H}^\bullet(\mathbf{V}_\lambda/K, (\mathcal{E}_\lambda, \nabla_\lambda)) = \mathbf{H}^\bullet(\hat{Y}_K, j_{\lambda *}\mathcal{DR}_\lambda))$$

we obtain the corollaries:

Corollary 2.5. *Under the assumptions of the theorem, the morphism*

$$\overline{\alpha} : \mathbf{H}^\bullet(X_K^{an}/K, (\mathcal{E}^{an}, \nabla^{an})) \longrightarrow H^\bullet_{rig}(X_\circ/K, (\mathcal{E}^\dagger, \nabla^\dagger))$$

is an isomorphism.

Corollary 2.6. *Under the assumptions of the theorem the morphism:*

$$\overline{\alpha} \circ \overline{\beta} : \mathbf{H}^\bullet(X_K/K, (\mathcal{E}, \nabla)) \longrightarrow H^\bullet_{rig}(X_\circ/K, (\mathcal{E}^\dagger, \nabla^\dagger))$$

is an isomorphism (cf. 2.1).

Corollary 2.6 is our comparison theorem between algebraic and the rigid cohomology with logarithmic coefficients.

§3 The hypotheses $(\mathbf{NL})_{\mathbf{G}}$ and $(\mathbf{SC})_{\mathbf{G}}$

We will illustrate by an example the role played by the two hyphoteses $(\mathbf{NL})_{\mathbf{G}}$ and $(\mathbf{SC})_{\mathbf{G}}$.

We consider the case $Y = \mathbf{P}^1_{\mathcal{V}}$, perfectly analogous to the one of Y = any proper smooth $\mathcal{V}$-scheme of relative dimension 1. We put: $D = D(0,1^-)$, $D^* = D \setminus \{0\}$. For $\lambda \in (0,1)$ we set $C_\lambda = \{x \in K \mid \lambda < |x| < 1\}$. So we now have:

$$\begin{array}{ccc} C_\lambda & \xrightarrow{j_\lambda} & D \\ \downarrow & \nearrow & \\ D^* & & \end{array}$$

For $\gamma \in K$ we consider the complexes:

$$\begin{array}{ccccccccc} a) & = & 0 & \longrightarrow & \mathcal{O}(D) & \xrightarrow{\nabla_\gamma} & \frac{1}{z}\Omega^1(D) & \longrightarrow & 0 \\ \downarrow & & & & \downarrow & & \downarrow & & \\ b) & = & 0 & \longrightarrow & \mathcal{O}(D^*) & \xrightarrow{\nabla_\gamma} & \frac{1}{z}\Omega^1(D^*) & \longrightarrow & 0 \\ \downarrow & & & & \downarrow & & \downarrow & & \\ c) & = & 0 & \longrightarrow & \mathcal{O}(C_\lambda) & \xrightarrow{\nabla_\gamma} & \frac{1}{z}\Omega^1(C_\lambda) & \longrightarrow & 0 \end{array}$$

where

$$\nabla_\gamma = d + \gamma \frac{dz}{z}.$$

Theorem 3.1.
i) *If $\gamma \in K$ is not a positive integer and $\rho(\gamma) = 1$, the inclusions:*

$$a) \hookrightarrow b) \hookrightarrow c)$$

are homotopy equivalences.

ii) *If $\rho(-\gamma) = 1$ then the complex $a)$ has finite dimensional cohomology.*

Proof. *i)* Let us consider for example $a) \xrightarrow{i} c)$. We construct a morphism of complexes $c) \xrightarrow{R} a)$:

$$R(\sum_{n\in\mathbf{Z}} a_n z^n) = \sum_{n\geq 0} a_n z^n$$

$$R(\sum_{n\in\mathbf{Z}} a_n z^n \frac{dz}{z}) = \sum_{n\geq 0} a_n z^n \frac{dz}{z}$$

which is the left inverse of i:

$$R \circ i = id_{a)}$$

while:

$$id_{c)} - i \circ R = \Delta_\gamma \circ H_\gamma + H_\gamma \circ \nabla_\gamma$$

where the homotopy operator H_γ :

$$H_\gamma \;:\; c) \to c)[-1]$$

is 0 in degrees 0,2, while:

$$H_\gamma(\sum_{n\in\mathbf{Z}} a_n z^n \frac{dz}{z}) = \sum_{n<0} \frac{a_n}{\gamma+n} z^n = (\sum_{n<0} a_n z^n) * (g_\gamma(z^{-1}) - \frac{1}{\gamma})$$

where " $*$ " denotes the Hadamard product with respect to z^{-1}.

Now $\sum_{n<0} a_n z^n$ is analytic for $|z| > \gamma$, while, by the hypothesis on γ, $g_\gamma(z^{-1})$ is analytic for $|z| > 1$. Thus H_γ takes its values in $\mathcal{O}(C_\lambda)$. The same argument works for the other inclusions.

ii) We define

$$H^\gamma \;:\; a) \longrightarrow a)[-1]$$

as

$$H^\gamma(\sum_{n\geq 0} a_n z^n \frac{dz}{z}) = \sum_{\substack{n=0\\ n\neq -\gamma}}^{\infty} \frac{a_n}{\gamma+n} z^n = -(\sum_{n\geq 0} a_n z^n) * (g_\gamma(z)).$$

If $\rho(-\gamma) = 1$, H^γ takes its values in $\mathcal{O}(D)$. If γ is not a negative integer nor 0,

$$id_{a)} = \nabla_\gamma H^\gamma + H^\gamma \nabla_\gamma$$

so that $a)$ is acyclic. If $\gamma = -n_\circ$, $n_\circ \in \{0, 1, \ldots\}$ then $a)$ contains the subcomplex

$$d) \quad = \quad 0 \to K z^{n_\circ} \longrightarrow K z^{n_\circ} \frac{dz}{z} \to 0\,,$$

$$d) \xrightarrow{j} a)\,.$$

The previous inclusion has an obvious retraction r

$$a) \xrightarrow{r} d) \xrightarrow{j} a)$$

which is the left inverse of j i.e.:

$$r \circ j = id_{d)},$$

while

$$id_{d)} - j \circ r = \nabla_\gamma \circ H^\gamma + H^\gamma \circ \nabla_\gamma.$$

Q.E.D.

Corollary 3.2. *Under the assumptions of the previous theorem the morphisms of complexes of abelian sheaves on D:*

$$\begin{array}{cccccccccc}
\tilde{a}) & = & 0 & \longrightarrow & \mathcal{O}_D & \xrightarrow{\nabla_\gamma} & \Omega^1_{D/K} < 0 > & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & \\
\tilde{b}) & = & 0 & \longrightarrow & j_* \mathcal{O}_{D^*} & \xrightarrow{\nabla_\gamma} & j_* \Omega^1_{D^*/K} & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & \\
\tilde{c}) & = & 0 & \longrightarrow & j_{\gamma *} \mathcal{O}_{C_\lambda} & \xrightarrow{\nabla_\gamma} & j_{\lambda *} \Omega^1_{C_\lambda/K} & \longrightarrow & 0
\end{array}$$

induce isomorphisms in hypercohomology:

$$\mathbf{H}^\bullet(D, \tilde{a})) \xrightarrow{\sim} \mathbf{H}^\bullet(D, \tilde{b})) \xrightarrow{\sim} \mathbf{H}^\bullet(D, \tilde{c})).$$

Proof. For any sheaf $\mathcal{F}$ which appears in the above diagram:

$$H^q(D, \mathcal{F}) = 0$$

for $q > 0$. (In fact D,D^* are quasi-Stein while j_λ is a quasi-Stein map). We are then reduced to prove quasi-isomorphisms for the complexes of global sections, thus to the theorem. **Q.E.D.**

We will show now discuss the role played by the hypothesis $(\mathbf{SC})_\mathbf{G}$.

Consider a system of linear differential equations of the form:

$$\mathcal{S}_G \qquad \frac{d}{dx} y = G(x) y$$

with $G(x) \in \mathcal{M}_n(K(x))$. Let $Z_K = \{a_1, \ldots, a_s\}$ be the set of singular points of $\mathcal{S}_G$ in $\hat{\mathbf{P}}^1_K$ and let $Z_\circ = \{\operatorname{sp} a_1, \ldots, \operatorname{sp} a_r\}$ where

$$\operatorname{sp} : \hat{\mathbf{P}}^1_K \longrightarrow \mathbf{P}^1_k$$

is the specialization map. Then $\mathcal{S}_G$ defines a connection on the sheaf $\mathcal{O}^n$ over $X_K^{an} = \hat{\mathbf{P}}^1_K \setminus Z_K$. When is this connection overconvergent along $Z_\circ$? If we intend to sick with the matrix G, we are forced to assume that $Z_\circ$ contains $x = \infty$; then, Berthelot's condition involves:

$$\mathbf{V}_\lambda = \hat{\mathbf{P}}^1_K \setminus \bigcup_{i=1}^{r} D(a_i, \lambda^+), \quad \lambda \in (0,1)$$

and the matrices giving the action of $(\frac{d}{dx})^m$ on solutions of $\mathcal{S}_G$:

$$(\frac{d}{dx})^m y = G^{(m)} y, \quad m \in \mathbf{N}$$

$(G^{(\circ)} = I_n,\ G^{(1)} = G,\ G^{(m+1)} = \frac{dG^{(m)}}{dx} + G^{(m)} G^{(1)})$. It is :

$(\mathbf{SC})_{\mathbf{G}}$ (Berthelot) For each $\eta \in (0,1)$ $\exists \lambda \in (0,1)$, such that

$$\lim_{m\to\infty} \|\frac{G^{(m)}}{m!}\|_{\mathbf{V}_\lambda} \eta^m = 0$$

(where $\| - \|_{\mathbf{V}_\lambda}$ denotes the supnorm on $\mathbf{V}_\lambda$, for matrices).

Recall the Gauss norm on $K(x)$: it is defined on $K[x]$ as:

$$|\sum a_i x^i|_{\mathcal{G}} = Sup|a_i|$$

and extended to an absolute value $| - |_{\mathcal{G}}$ of $K(x)$ by multiplicativity. It depends essentially on the $\mathcal{V}$-structure of $\hat{\mathbf{P}}^1_K$: if a function $f \in K(x)$ has no poles in an open disk $\mathcal{D}$ of radius 1, then

$$|f|_{\mathcal{G}} = \|f\|_{\mathcal{D}}.$$

So, condition $(\mathbf{SC})_{\mathbf{G}}$ implies:

$(\mathbf{SC})'_{\mathbf{G}}$, $\forall \eta \in (0,1)$:

$$\lim_{m\to\infty} |\frac{G^{(m)}}{m!}|_{\mathcal{G}} \eta^m = 0.$$

Condition $(\mathbf{SC})'_{\mathbf{G}}$ appears in the work of Dwork, Robba, Christol, André ([An1], [An2], [Ch], [Ch-Dw]), as that of convergence of the solutions of $\mathcal{S}_G$ in the generic disk of radius 1. This is motivated from the fact that the matrix function:

$$\mathcal{U}_{G,t}(x) = \sum_{m=0}^{\infty} \frac{G^{(m)}(t)}{m!}(x-t)^m$$

is a fundamental solution matrix of $\mathcal{S}_G$ at a generic unit t and:

$$|\frac{G^{(m)}(t)}{m!}| = |\frac{G^{(m)}}{m!}|_{\mathcal{G}}.$$

Let's now check that $(\mathbf{SC})'_{\mathbf{G}} \Rightarrow (\mathbf{SC})_{\mathbf{G}}$. Let us put, for each $\lambda \in (0,1)$,

$$]X_\circ[=)W = \hat{\mathbf{P}}^1_K \setminus \bigcup_{i=1}^{r} D(a_i, 1^-) \subset \mathbf{V}_\lambda\ .$$

Then $(\mathbf{SC})'_{\mathbf{G}}$ certainly implies:

$$\lim_{m\to\infty} \|\frac{G^{(m)}}{m!}\|_{W} \eta^m = 0$$

for each $\eta \in (0,1)$. So we are left to consider separated annuli around the singular points. We may assume:

$$a_1, \dots, a_d \in D = D(0,1^-)\,,\ a_{d+1}, \dots, a_r \notin D.$$

Let

$$f(x) = \prod_{i=1}^{d}(x-a_i)^N$$

be such that $fG \in \mathcal{M}_n(\mathcal{O}(D))$. Then also $f^m G^{(m)} \in \mathcal{M}_n(\mathcal{O}(D)$, for each $m \in \mathbb{N}$. For $\lambda > \max_{i=1,\dots,d}|a_i|$ we have:

$$\|\frac{G^{(m)}}{m!}\|_{C_\lambda} \leq \|f^m\frac{G^{(m)}}{m!}\|_{C_\lambda}\|f^{-m}\|_{C_\lambda} \leq |f^m\frac{G^{(m)}}{m!}|_{\mathcal{G}}\lambda^{-mNd} = |\frac{G^{(m)}}{m!}|_{\mathcal{G}}\lambda^{-mNd},$$

since $|f|_{\mathcal{G}} = 1$. So, if λ is also $> \eta^{\frac{1}{Nd}}$, one has $\eta/\lambda^{Nd} \in (0,1)$ and

$$(\mathbf{SC})'_{\mathbf{G}} \qquad \lim_{m\to\infty}|\frac{G^{(m)}}{m!}|_{\mathcal{G}}(\frac{\eta}{\lambda^{Nd}})^m = 0$$

implies

$$(\mathbf{SC})_{\mathbf{G}} \qquad \lim_{m\to\infty}\|\frac{G^{(m)}}{m!}\|_{C_\lambda}\eta^m = 0.$$

So:

Proposition 3.3. *The system S_G is overconvergent along its polar divisor iff its solutions converge in the generic disk of radius one.*

What we said should justify the following weaker, local, condition on a system on $D = D(0,1^-)$:

$$\mathcal{L}_G \qquad x\frac{d}{dx}y = Gy$$

with $G \in \mathcal{M}_n(\mathcal{O}(D))$. We put, as before,

$$x^m(\frac{d}{dx})^m y = G_m y$$

for $m \in \mathbb{N}$. We consider the condition:

$(\mathbf{SC})_{\mathbf{L}}$ $\forall\eta \in (0,1)$, for each affinoid $V \subset D$

$$\lim_{m\to\infty}\|\frac{G_m}{m!}\|_V\eta^m = 0$$

We also define the *type* of the system $\mathcal{L}_G$ at 0 as:

$$\rho = \prod \rho(\gamma)^{e_\gamma} \qquad (\in [0,1])$$

if $\det(x - G(0)) = \prod(x-\gamma)^{e_\gamma}$.

Our main result in this framework is:

Theorem 3.4. *Assume the system* $\mathcal{L}_G$, $G \in \mathcal{M}_n(\mathcal{O}(D))$ *satisfies condition* $(\mathrm{SC})_{\mathrm{L}}$. *Let* ρ *be the type of* $\mathcal{L}_G$ *at 0. Then any formally meromorphic column solution* y *of* $\mathcal{L}_G$ *at 0, is p-adically meromorphic for* $|x| < \rho$.

Corollary 3.5. *Under the assumptions of the theorem, assume also that the eigenvalues of* $G(0)$ *are p-adically non-Liouville. Then, the formally meromorphic solutions* y *of* L_G *at 0 are meromorphic in* D.

Consider the condition:

$(\mathrm{NL})_{\mathrm{L}}$ The additive subgroup Λ of K generated by the eigenvalues of $G(0)$, consists of p-adically non-Liouville numbers.

Corollary 3.6. *Assume conditions* $(\mathrm{NL})_{\mathrm{L}}$ *and* $(\mathrm{SC})_{\mathrm{L}}$ *hold for* $\mathcal{L}_G$. *For* $\nabla_G = d + G\frac{d}{dx}$, *consider the diagram of abelian sheaves on* D:

$$
\begin{array}{ccccccccc}
a) & = & 0 & \longrightarrow & \mathcal{O}_D^n[\frac{1}{x}] & \xrightarrow{\nabla_G} & \Omega^1_{D/K} \otimes \mathcal{O}_D^n[\frac{1}{x}] & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & \\
b) & = & 0 & \longrightarrow & j_*\mathcal{O}^n_{D^*} & \xrightarrow{\nabla_G} & j_*(\Omega^1_{D^*/K} \otimes \mathcal{O}^n_{D^*}) & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & \\
c) & = & 0 & \longrightarrow & j_{\lambda *}\mathcal{O}^n_{C_\lambda} & \xrightarrow{\nabla_\gamma} & j_{\lambda *}(\Omega^1_{C_\lambda/K} \otimes \mathcal{O}^n_{D^*}) & \longrightarrow & 0
\end{array}
$$

The morphisms $a) \hookrightarrow b) \hookrightarrow c)$ *induce isomorphisms of hypercohomology groups over* D.

Proof. It consists of the following steps:

1) Use the formal theory of logarithmic systems to formally reduce to upper-triangular systems.

2) Use corollary 3.5 to show that the formal equivalence referred to in step 1), is in fact convergent on D.

So, we may assume that G is upper-triangular.

3) Reduce to systems of rank 1, via the spectral sequence of filtered complexes.

4) Apply the corollary to theorem 3.1, after translating the exponents by an integer, if needed (multiplication by x^N is an isomorphism). **Q.E.D.**.

We now come to our main result:

Corollary 3.7. *Theorem 2.4 and its corollaries hold for* $Y = \mathbf{P}^1$.

Proof. Recall that in theorem 2.4 we fixed a $\lambda \in (0,1)$ and that we had:

$$\mathbf{V}_\lambda = \hat{\mathbf{P}}^1_K \setminus \bigcup_{i=1}^{r} D(a_i, \lambda^+) \xrightarrow{j_\lambda} \hat{\mathbf{P}}^1_K.$$

We deal with a connection $(\mathcal{E}, \nabla)$ with logarithmic singularities at $a_1, \ldots, a_r$ (assumed to lie in distinct residues classes).

We are supposed to examine:

$$\mathcal{F}^\bullet = j_*\mathcal{DR}(X_K^{an}/K, (\mathcal{E}^{an}, \nabla^{an}) \xrightarrow{i} \mathcal{F}_\lambda^\bullet = j_{\lambda *}\mathcal{DR}(\mathbf{V}_\lambda/K, (\mathcal{E}_\lambda, \nabla_\lambda).$$

We choose $\eta \in (\lambda, 1)$ and consider the admissible covering $\mathcal{W}$ of $\hat{\mathbf{P}}_K^1$:

$$\mathcal{W} = \{\mathbf{V}_\eta, D(a_1, 1^-), \dots, D(a_r, 1^-)\}.$$

We have that the restriction:

$$i_{|\mathbf{V}_\eta} : \mathcal{F}^\bullet_{|\mathbf{V}_\eta} \simeq \mathcal{F}^\bullet_{\lambda|\mathbf{V}_\eta}$$

is the identity, while on each disk $D(a_i, 1^-)$ we are in the situation of corollary 3.6.
Our result follows from the spectral sequence of hypercohomology associated to the covering $\mathcal{W}$:

$$E_2^{p,q}(\mathcal{F}^\bullet) = H^p(\mathcal{W}, h^q(\mathcal{F}^\bullet)) \Rightarrow \mathbf{H}^\bullet(\hat{\mathbf{P}}_K^1, \mathcal{F}^\bullet)$$

where $h^q(\mathcal{F}^\bullet)$ denotes the presheaf:

$$U \longmapsto \mathbf{H}^q(U, \mathcal{F}^\bullet)$$

(and similarly for $\mathcal{F}_\lambda^\bullet$).
The morphism of spectral sequences:

$$E_\bullet^{\bullet,\bullet}(\mathcal{F}^\bullet) \longrightarrow E_\bullet^{\bullet,\bullet}(\mathcal{F}_\lambda^\bullet)$$

is in fact an isomorphism at the E_2 level. (In fact, since

$$H^q(U, \mathcal{G}) = 0 \quad \forall q > 0$$

for any open set U of the nerve of $\mathcal{W}$ and any sheaf $\mathcal{G}$ under consideration, the Čech bicomplexes of $\mathcal{W}$ with coefficients in $\mathcal{F}^\bullet$ and $\mathcal{F}_\lambda^\bullet$, actually calculate the hypercohomology of $\hat{\mathbf{P}}_K^1$). **Q.E.D.**

§4 Hints for the general case.

We point out some useful facts about the general case.

I) *Existence of tubular neighborhoods of radius* 1 *of* $\hat{Z}_K$ *in* $\hat{Y}_K$ ([Ba-Ct3]).

We may refine the covering of $\hat{Y}_K$

$$\hat{\mathcal{U}} = \{\hat{U}_K\}_{U \in \mathcal{U}}$$

obtained from the original $\mathcal{U}$. This will be done in connection with a given $\mathbf{V}_\lambda$, $\lambda \in (0,1)$, as in theorem 2. So, let $U \in \mathcal{U}$ be as in section 2, with coordinates $(x_1, \ldots, x_m)$ and assume the branches of Z_K meeting U_K are $Z_K^{(1)}, \ldots, Z_K^{(\nu)}$ of equation, resp., $x_1 = 0, \ldots x_\nu = 0$.

Let $\mathcal{T}_U = \{1, \ldots, \nu\}$, $\mathcal{S} \subset \mathcal{T}_U$. For $\eta \in (\lambda, 1)$, we put:

$$\mathbf{U}_{\mathcal{S},\eta} = \{p \in \hat{U}_K : |x_i(p)| < 1 \text{ if } i \in \mathcal{S} \text{ and } |x_i(p)| \geq \eta, \text{ if } i \in \mathcal{T}_U \setminus \mathcal{S}\}$$

the main point is:

Proposition 4.1. $\mathbf{U}_{\mathcal{S},\eta}$ *is a trivial bundle in open unit polydisks of relative dimension* $s = \mathcal{S}$ *over a smooth affinoid space* $V_{\mathcal{S},\eta} = SpmA_{\mathcal{S},\eta}$.

Proof. It is an immediate consequence of the following:

Lemma 4.2. *Let* S, Z, P *be formal* $\mathcal{V}$*-schemes of finite presentation and*

$$\begin{array}{ccc} Z & \hookrightarrow & P \\ \downarrow & \nearrow & \\ S & & \end{array}$$

be a closed immersion of S*-objects where* $Z \longrightarrow S$ *(resp.* $P \longrightarrow S$*) is smooth of relative dimension* d *(resp.* $d + s$*). Assume (always true locally on* P*) that* $S = SpfA$, $P = SpfB$, $Z = SpfC$ *are affine and that* $C = B/J$ *where* $J = (f_1, \ldots, f_s)$ *is generated by* s *elements. Then, if* $i : Z_K \hookrightarrow]Z_\circ[_P$ *denotes the closed immersion, there exists a retraction* $\sigma :]Z_\circ[_P \rightarrow Z_K$ *and an isomorphism:*

$$]Z_\circ[_P \xrightarrow{\sim} Z_K \times D^s$$

such that the diagram:

$$\begin{array}{ccc} Z_K & \xrightarrow{(id_{Z_K}, 0)} & Z_K \times D^s \\ \downarrow i & \sim \nearrow & \downarrow pr_1 \\ ]Z_\circ[_P & \xrightarrow{\sigma} & Z_K \end{array}$$

commutes.

Q.E.D.

II) *The formal and convergent theory of systems with logarithmic singularities on standard spaces* ([Ba-Ct2]).

Here A is regular Tate K-algebra with no zero-divisors, and we define

$$D^s_A = SpmA \times D^s.$$

We consider systems of P.D.E.'s of the form:

$$\mathcal{L}_G \qquad \begin{aligned} \partial y &= G_\partial y \qquad & \partial \in Der^c(A/K) \\ x^\alpha (\frac{\partial}{\partial x})^\alpha y &= G_\alpha y \qquad & \alpha \in \mathbf{N}^s \end{aligned}$$

where $G_\partial, G_\alpha \in \mathcal{M}_n(\mathcal{O}(D^s_A))$ satisfy the usual integrability conditions.

We consider conditions $(\mathbf{NL})_\mathbf{L}$ and $(\mathbf{SC})_\mathbf{L}$ on $\mathcal{L}_G$. We then develop a refined formal (i.e. on $A[\![x_1, \ldots, x_s]\!]$) and convergent (i.e. on $\mathcal{O}(D^s_A)$) theory of such systems, analogous to the one for ordinary systems.

References.

[A1] André Y. : " Spécialisation dans les disques singuliers réguliers", in Groupe d'Etude d'Analyse Ultramétrique 1985/86; Y.Amice, G.Christol, P.Robba, Eds. ; Publ. Math. Université Paris VII.

[A2] André Y. : "G-functions and Geometry", Vieweg-Verlag, Bonn 1989.

[Ba1] Baldassarri F. : " Comparaison entre la cohomologie algébrique et la cohomologiep-adique rigide à coefficients dans un module différentiel I" . Inv. Math. 87(1987), 83-99 .

[Ba2] Baldassarri F. : " Comparaison entre la cohomologie algébrique et la cohomologiep-adique rigide à coefficients dans un module différentiel II". Math. Ann.280 (1988), 417-439.

[Ba-Ct0] Baldassarri F.,Chiarellotto B. :" On Christol's theorem. A generalization to systems of PDE's with logarithmic singularities depending upon parameters", to appear in *Proceedings of the Special Session on Algebraic Geometry and Number Theory*, AMS Publications, Series of Contemporary Mathematics.

[Ba-Ct1] Baldassarri F.,Chiarellotto B. :" On André's transfert theorem.", to appear in *Proceedings of the Special Session on Algebraic Geometry and Number Theory*, AMS Publications, Series of Contemporary Mathematics.

[Ba-Ct2] Baldassarri F.,Chiarellotto B. :" Formal and p-adic theory of differential systems with logarithmic singularities depending upon parameters", article in preparation.

[Ba-Ct3] Baldassarri F.,Chiarellotto B. :" Algebraic versus rigid cohomology with logarithmic coefficients.", to appear in *Proceedings of the Barsotti Memorial Symposium. Abano Terme (Padova), June 24-27, 1991*, Perspectives in Mathematics, Academic Press.

[Ber] Berthelot P. : "Cohomologie rigide et cohomologie rigide á support propre". A paraitre dans Astérisque.

[BGR] Bosch S.; Guntzer U.; Remmert R. : "Non archimedean Analysis". Grundlehren der Math. Wissenschaften 261, Springer–Verlag, Berlin 1984.

[Ch] Christol G.: "Un théorème de transfert pour les disques singuliers réguliers", Astérisque 119-120 (1984),151-168.

[Ch-Dw] Christol G., Dwork B. : " Effective p-adic bounds at regular singular points", Duke Math. J. Vol. 62 (1991), 689-720.

[Ct] Chiarellotto B. :" A comparison theorem in p-adic cohomology". Ann. di Mat. Pura e Appl. (IV) 153 (1988), 115-132 .

[K] Kiehl R. :" Die de Rham kohomologie algebraischer Mannigfaltigkeiten uber einem bewerteten Korper". Publ. Math. I.H.E.S. 33 (1967) ,5-20.

[Ra] Raynaud M "Géométrie analitique rigide". Bull. Soc. Math. France, Mémoire 39-40 (1974), 319- 327.

[SGAIV] "Seminaire de Geometrie Algebrique. Theorie des Topos et Cohomologie etale des Schémas". M. Artin; A. Grothendieck L.N. in Math., 269–270 Springer-Verlag Berlin-New York 1972.

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