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Asymptotic Completeness for 3–Particle Stark Hamiltonian

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In the present note we make a brief review on the asymptotic completeness for three–particle Stark Hamiltonian, which has recently been proved by the author [11].

Consider a system of three particles moving in a uniform electric field $\mathcal{E} \in R^3$. The total energy Hamiltonian (Schrödinger operator) for such a system takes the following form :

$$H_{tot} = -\sum_{j=1}^{3}(\Delta/2m_j + e_j\langle\mathcal{E}, r_j\rangle) + \sum_{1\le j<k\le 3} V_{jk}(r_j - r_k),$$

where m_j, e_j and $r_j \in R^3$, $1 \le j \le 3$, denote the mass, charge and position vector of the j–th particle respectively, while $-e_j\langle\mathcal{E}, r_j\rangle$, $\langle\ ,\ \rangle$ being the usual scalar product, is the energy of interaction with the electric field and the real function V_{jk} is the potential interaction between the j–th and k–th particles.

During the last decade, many remarkable works [2, 3, 4, 7, 9, 10] have made major progress in the scattering theory of many particles in the absence of electric fields. Among these works, Sigal–Soffer [10] first proved the asymptotic completeness of wave operators for N–body scattering systems with a large class of short–range pair potentials, and now much attention is paid to the long–range scattering cases, including the Coulomb scattering system. On the other hand, the spectral and scattering theory of one(two)– particle systems in the presence of electric field has also been studied by many authors [1, 5, 8, 13]. However, there seems to be only a few works for the case of many–particle systems. Korotyaev [6] has proved the asymptotic completeness for three–particle scattering systems by making use of the Faddeev equation method. We here present a different proof, which is based on the local commutator method and on the propagation estimate showing that the relative motion of particles is asymptotically concentrated on classical trajectories. We do not necessarily assume that a two-particle subsystem Hamiltonian with zero reduced charge does not have a zero energy resonance. This improves the results obtained by [6], although we have to impose a somewhat restrictive smoothness assumption on the pair potentials V_{jk}.

§1. Asymptotic completeness

We proceed to the precise formulation of the obtained results. We start by making the assumption on the pair potentials V_{jk}.

$(V)_\rho$ $V_{jk}(y)$, $y \in R^3$, is a real C^2-smooth function with the following decay properties as $|y| \to \infty$:

$(V.0)$ $V_{jk}(y) = O(|y|^{-\rho})$ for some $\rho > 1/2$;

$(V.1)$ $\partial_y^\alpha V_{jk}(y) = o(1)$, $|\alpha| = 1$;

$(V.2)$ $\partial_y^\alpha V_{jk}(y) = O(1)$, $|\alpha| = 2$.

Next we remove the center-of-mass motion. For notational brevity, we here assume that the masses of all the three particles equal and take the value one ;

$$m_j = 1, \qquad 1 \le j \le 3.$$

For such a system, the configuration space X in the center-of-mass frame is given as

$$X = \{r = (r_1, r_2, r_3) \in R^{3\times 3} : \sum_{j=1}^{3} r_j = 0\}$$

and the energy Hamiltonian H takes the form

$$H = -\frac{1}{2}\Delta - \langle E_X, r\rangle + V \quad \text{on } L^2(X),$$

where E_X denotes the projection onto X of $E = (e_1\mathcal{E}, e_2\mathcal{E}, e_3\mathcal{E}) \in R^{3\times 3}$. We also assume that

$$E_X \neq 0.$$

Under assumption $(V)_\rho$, the operator H formally defined above admits a unique self-adjoint realization in $L^2(X)$. We denote by the same notation H this self-adjoint realization.

Let $P_H : L^2(X) \to L^2(X)$ be the eigenprojection associated with H. Roughly speaking, the problem of asymptotic completeness is to determine completely the asymptotic states as $t \to \pm\infty$ of solutions to the Schrödinger equation

$$i\,\partial_t u = H\,u, \quad u(0) = \psi \in Range\,(Id - P_H).$$

Such asymptotic states are characterized by the range of wave operataors.

To formulate presicely the asymptotic states above, we continue to introduce new notations. Let $a = \{(j,k), l\}$, $j < k$, be a 2–cluster decomposition of the set $\{1,2,3\}$. Then we define the two subspaces of X as follows :

$$\begin{aligned} X^a =& \{r = (r_1, r_2, r_3) \in X : r_j + r_k = 0\}, \\ X_a =& \{r = (r_1, r_2, r_3) \in X : r_j = r_k\}. \end{aligned}$$

As is easily seen, these two spaces are mutually orthogonal and span the total space X, so that $L^2(X)$ is decomposed as

$$L^2(X) = L^2(X^a) \otimes L^2(X_a).$$

Let $\pi^a : X \to X^a$ and $\pi_a : X \to X_a$ be the projection onto X^a and X_a, respectively. For a generic point $x \in X$, we write $x^a = \pi^a\,x$ and $x_a = \pi_a\,x$. We also write V_a for the pair potential V_{jk}. We further define the cluster Hamiltonian H_a as

$$H_a = -\frac{1}{2}\,\Delta - \langle E_X, r\rangle + V_a(r_j - r_k) \quad \text{on} \quad L^2(X).$$

According to the tensor product decomposition above, this operator has the following decomposition :

$$H_a = H^a \otimes Id + Id \otimes T_a \quad \text{on} \quad L^2(X^a) \otimes L^2(X_a),$$

where

$$\begin{aligned} H^a =& -\frac{1}{2}\,\Delta - \langle E^a, r\rangle + V_a, \quad E^a = \pi^a\,E_X, \\ T_a =& -\frac{1}{2}\,\Delta - \langle E_a, r\rangle, \quad E_a = \pi_a\,E_X. \end{aligned}$$

If $a = \{(1),(2),(3)\}$ is a 3–cluster decomposion, the operator H^a is defined as the zero operator acting on $L^2(X^a) = C$ (scalar field), so that H_a becomes the free Stark Hamiltonian

$$H_0 = -\frac{1}{2}\Delta - \langle E_X, r\rangle \quad \text{on} \quad L^2(X).$$

If $E^c = 0$ for some 2–cluster decomposition c, then the two–particle subsystem Hamiltonian H^c with zero reduced charge $E^c = 0$ has in general bound states and hence scattering channels associated with such bound states may arise even in a three–particle system with electric field as in the case of the absence of electric field. Thus the consideration is divided into the following two cases :

Case (i) $E^a \neq 0$ for all 2–cluster decompositions ;

Case (ii) $E^c = 0$ for some 2–cluster decomposition c.

We should note that there exists at most one 2–cluster decomposition c with $E^c = 0$, which follows immediately from the assumption $E_X \neq 0$.

Finally we introduce the wave operators. We define $W_0^{\pm} : L^2(X) \to L^2(X)$ by

$$W_0^{\pm} = s - \lim_{t\to\pm\infty} \exp(itH)\exp(-itH_0).$$

Let c be as in Case (ii) and let $P^c : L^2(X^c) \to L^2(X^c)$ be the eigenprojection associated with H^c. Then we further define $W_c^{\pm} : L^2(X) \to L^2(X)$ by

$$W_c^{\pm} = s - \lim_{t\to\pm\infty} \exp(itH)\exp(-itH_c)(P^c \otimes Id).$$

If the wave operators $W_0^{\pm}$ and $W_c^{\pm}$ exist, then these operators can be easily proved to have the following properties : (i) their ranges are closed in $L^2(X)$ and are contained in $Range\,(Id - P_H)$; (ii) their ranges are orthogonal to each other ; $Range\, W_0^{\pm} \perp Range\, W_c^{\pm}$.

With these notations, we are now in a position to formulate the main theorem.

THEOREM (ASYMPTOTIC COMPLETENESS). *Let the notations be as above. One has the following two statements.*

(1) Consider Case (i). Assume $(V)_\rho$ with $\rho > 1/2$. Then the wave operators $W_0^{\pm}$ exist and are aymptotically complete ;

$$Range\, W_0^{\pm} = Range\,(Id - P_H).$$

(2) Consider Case (ii) and let c be the 2–cluster decomposition as in Case (ii). Assume $(V)_\rho$ with $\rho > 1$. Then the wave operators $W_0^{\pm}$ and $W_c^{\pm}$ exist and are asymptotically complete ;

$$Range\, W_0^{\pm} \oplus Range\, W_c^{\pm} = Range\,(Id - P_H).$$

We end the section by making several comments on the main theorem above.

REMARK 1. Under assumption $(V)_\rho$ with $\rho > 3/4$, the three-particle Stark Hamiltonian H can be shown to have no eigenvalues, so that $Range\,(Id - P_H) = L^2(X)$ ([12]).

REMARK 2. In statement (2), the decay assumption $V_a(y) = O(|y|^{-\rho})$, $\rho > 1$, for $a \neq c$, c being as in Case (ii), is used to prove only the existence of wave operators $W_c^{\pm}$. If we assume that the zero eigenstate $\varphi \in L^2(X^c)$ of two-particle subsystem Hamiltonian H^c with zero reduced charge has the decaying property $(1 + |x^c|)^\nu \varphi(x^c) \in L^2(X^c)$ for some $\nu > 1/2$, then statement (2) can be proved to remain true under the weak decay assumption $V_a(y) = O(|y|^{-\rho})$, $\rho > 1/2$, for $a \neq c$.

REMARK 3. As stated above, the asymptotic completeness for three-particle scattering systems in electric fields has been proved in Korotyaev [6] by use of the Faddeev equation method. In this work, the operator H^c with zero reduced charge is assumed to have no zero energy resonances and also the pair potential V_c is assumed to satisfy the stronger decay assumption $(V)_\rho$ with $\rho > 2$. Thus statement (2) above improves slightly the results obtained by [6], although the additional smoothness assumptions are imposed on the pair potentials in the present work.

§2. Local commutator method and propagation estimates

Let $\sigma_p(H)$ be the set of point spectrum of H. One can prove that $\sigma_p(H)$ forms a discrete set in R^1 with possible accumulating points $\pm\infty$. We now fix arbitrarily a real function $g \in C_0^\infty(R^1)$ with $supp\, g \cap \sigma_p(H) = \emptyset$. The proof of the main theorem is done by showing that the scattering state $\exp(-itH)g(H)\psi$, $\psi \in L^2(X)$, is aymptotically concentrated on classical trajectories.

Let S_X be the unit sphere in X. We write E_X as

$$E_X = E_0\,\omega, \quad E_0 = |E_X| > 0, \quad \omega \in S_X,$$

and also $x \in X$ as

$$x = z\,\omega + z_\perp, \quad z \in R^1, \quad z_\perp \in \Pi_\omega,$$

where Π_ω stands for the hyperplane orthogonal to ω. As is easily seen, charged classical particles are scattered along the direction ω of the electric field and the coordinate z asymptotically takes the value $z \sim E_0 t^2/2$ as $t \to \pm\infty$ along these classical trajectories. These facts imply the following two propagation estimates for the above scattering state $\exp(-itH)g(H)\psi$:

$$\int \|\langle x\rangle^{-\nu} \exp(-itH)g(H)\psi\|^2_{L^2(X)}\, dt \le C\,\|\psi\|^2_{L^2(X)}, \quad \nu > 1/4,$$

$$\int \|\langle x\rangle^{-1/4} q(x) \exp(-itH)g(H)\psi\|^2_{L^2(X)}\, dt \le C\,\|\psi\|^2_{L^2(X)},$$

where q is a bounded function vanishing in a conical neighborhood of the direction ω. These two estimates are derived on the basis of local commutator method.

LEMMA (LOCAL COMMUTATOR ESTIMATE). *Let $\lambda \notin \sigma_p(H)$ be fixed and let $g \in C_0^\infty(R^1)$ be a real function supported in a small neighborhood around λ. Then, for any $\delta > 0$ small enough, one can take the support of g so small that*

$$g(H)\, i[H, A]\, g(H) \ge (E_0 - \delta) g(H)^2$$

in the form sense, where $[\ ,\ ]$ denotes the commutator relation and the operator A is defined as $A = \frac{1}{i}\, \omega \cdot \nabla_x$.

The lemma above may be intuitively understood as follows. Consider the quantity

$$\langle z\rangle_t = (z \exp(-itH)g(H)\psi, \exp(-itH)g(H)\psi)_{L^2(X)}.$$

This quantity behaves like $\langle z\rangle_t \sim E_0 t^2/2$ as $t \to \pm\infty$ and hence it follows that

$$\frac{d^2}{dt^2}\langle z\rangle_t = (g(H) i[H, A] g(H) \exp(-itH)\psi, \exp(-itH)\psi)_{L^2(X)} \sim E_0.$$

In fact, the relation $i[H_0, A] = E_0$ exactly holds for the free Stark Hamiltonian H_0. The local commutator method was initiated by Mourre [7] and the commutator estimate as in the lemma above is called the Mourre estimate. This method has played an important role in the spectral and scattering theory for many–particle Schrödinger operators without electric fields for which the operator A is taken as $A = \frac{1}{2i}(x \cdot \nabla_x + \nabla_x \cdot x)$.

In general, Stark Hamiltonians take all the real numbers as possible values of energies. For example, the two–particle subsystem Hamiltonians H^a and T_a, a being a 2–cluster decomposition, can take all the real numbers as energies, even if the energy of three–particle Hamiltonian H under consideration is localized in a bounded interval. This is not the case for Hamiltonians without uniform electric fields, because such Hamiltonians are bounded below. This is one of main differences between the cases with electric fields and without electric fields, and also this difference makes it difficult to prove the local commutator estimate in the lemma above.

We shall explain this difficulty briefly. Now, let $f \in C_0^\infty(X^a)$ for 2–cluster decomposition a. To prove the above lemma, we have to show that

$$(2.1) \qquad \|f\, g(H_a)\, f\|_{L^2(X)\to L^2(X)} \to 0, \qquad supp\, g \to \{\lambda\}.$$

By use of the spectral representation for the two–particle free Stark Hamiltonian T_a, this operator can be represented as the direct integral ;

$$f\, g(H_a)\, f = \int \oplus f\, g(\theta + H^a)\, f\, d\theta.$$

By use of the Stone formula, the integrand is further represented as

$$f\, g(\theta + H^a)\, f = \frac{1}{2\pi i}\int g(\theta+\mu)\{R(\mu + i0; H^a) - R(\mu - i0; H^a)\}\, d\mu,$$

where $R(\zeta; H^a)$ denotes the resolvent of H^a. If $f\, R(\mu \pm i0; H^a)\, f$ is proved to be bounded uniformly in $\mu \in R^1$ as an operator from $L^2(X^a)$ into itself, then (2.1) follows at once. Thus we have to study the resolvent estimates at high energies for the two–particle subsystem Hamiltonians H^a.

LEMMA (RESOLVENT ESTIMATE AT HIGH ENERGIES). *The two–particle subsystem Hamiltonian H^a with $E^a \neq 0$ satisfies the resolvent estimate*

$$\|f\, R(\mu \pm i0; H^a)\, f\|_{L^2(X^a)\to L^2(X^a)} = O(1), \quad \mu \to \pm\infty.$$

This lemma plays a central role in proving the local commutator estimate. To prove the asymptotic completeness for N–particle scattering systems, we have to verify the resolvent estimates at high energies as above for all subsystem Hamiltonians. However, it does not seem to be easy to prove such estimates for many–particle systems.

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