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ON A ZETA FUNCTION FOR EQUIVALENCE CLASSES OF BINARY QUADRATIC FORMS

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0. INTRODUCTION

For an arbitrary number field $\mathbf{K}$ with ring of integers denoted by $\mathcal{O}$ we study $\mathrm{GL}(2,\mathcal{O})$ (=:G)– and $\mathrm{SL}(2,\mathcal{O})$–equivalence classes of binary quadratic forms $\Phi(x,y) = ax^2+bxy+cy^2$ defined over $\mathcal{O}$. After fixing $\Delta \in \mathcal{O}$ we define the following zeta functions for G–equivalence classes of the binary quadratic forms over $\mathcal{O}$ of discriminant Δ. For this we set $\mathrm{E}(\Phi) := \{g \in \mathrm{G} : (g\Phi) = \Phi\}$. We define

$$\zeta_\Delta(s) := \sum_{\substack{[\Phi] \\ \Delta(\Phi)=\Delta}} \sum_{\substack{(x,y)\in(\mathcal{O}\times\mathcal{O})/\mathrm{E}(\Phi) \\ (x,y)_{\mathcal{O}}=\mathcal{O} \\ \Phi(x,y)\neq 0}} |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Phi(x,y))|^{-s},$$

the first sum running over the G–equivalence classes of binary quadratic forms of discriminant Δ and the inner sum over pairs of numbers in $\mathcal{O}$ modulo the automorphism group of the form, which are coprime.

For the rational numbers and imaginary quadratic fields one can define this function also for $\mathrm{SL}(2,\mathcal{O})$–equivalence. It arises in the calculation of the Selberg trace formula for integral operators on $\mathrm{L}^2(\mathrm{PSL}(2,\mathcal{O})\backslash\mathbf{H})$ where $\mathbf{H}$ is either the two dimensional or the three dimensional upper half space and $\mathcal{O}$ is the rational integers or the ring of integers of an imaginary quadratic number field respectively, cf. [Z3,Ba2]. It turns out that G–equivalence is the right equivalence to generalize the wellknown formula for the field of rational numbers with $\mathrm{Sl}(2,\mathcal{O})$–equivalence.

We will express $\zeta_\Delta(s)$ closed form in terms of L–series for $\mathbf{K}$ by generalizing the proof for the rational integers (cf. [L],[Hi–Z]), which is based on counting the solutions of the congruence $b^2 \equiv \Delta(4a)$ in $\mathbf{Z}/2a\mathbf{Z}$, to arbitrary number fields. The final result is stated in theorem 4.1. Most proofs will be ommitted. They can be found in [Ba2].

1. PRELIMINARIES

Let $\mathbf{K}$ be an algebraic number field, $\mathcal{O}$ its ring of integers. For $a, b, c \in \mathcal{O}$ define the binary quadratic form

$$\Phi(x,y) = ax^2 + bxy + cy^2.$$

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It has discriminant $\Delta(\Phi) = b^2 - 4ac$. We can assign the symmetric matrix $A = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$ to Φ and write $\Phi(x,y) = (x,y)A(x,y)^t$.

On such a form the 2×2–matrices with integral coefficients operate by

$$\begin{aligned}[g]\Phi(x,y) := \Phi((x,y)g) &= (x,y)gAg^t(x,y)^t, \qquad g = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in \mathrm{M}(2,\mathcal{O}),\\ &= a'x^2 + b'xy + c'y^2\end{aligned}$$

with

$$\begin{aligned} a' &= a\alpha^2 + b\alpha\gamma + c\gamma^2 = \Phi(\alpha,\gamma)\\ b' &= 2a\alpha\beta + b(\alpha\delta + \beta\gamma) + 2c\gamma\delta \\ c' &= a\beta^2 + b\beta\delta + c\delta^2 = \Phi(\beta,\delta).\end{aligned} \tag{1.1}$$

The discriminant behaves under this operation like $\Delta([g]\Phi) = \det(g)^2\Delta(\Phi)$.

For the study of equivalence classes of binary quadratic forms of a fixed discriminant we introduce the following equivalence relation. We fix $\Delta \in \mathcal{O}$ and set $G := \mathrm{GL}(2,\mathcal{O})$.

1.1. Definition. *We call two binary quadratic forms Φ and Ψ over $\mathcal{O}$ of discriminant Δ G–equivalent if there exists an element $g \in \mathrm{G}$ such that*

$$(g\Phi)(x,y) := \det(g)^{-1}\Phi((x,y)g) = \Psi(x,y).$$

If we only allow transformations of determinant 1, *we call the forms 1–equivalent.*

This is in fact an equivalence relation. Two equivalent forms represent up to a unit the same numbers.

The matrices which leave a form invariant,

$$\mathrm{E}(\Phi) := \{g \in \mathrm{G} : (g\Phi) = \Phi\} \quad \text{and} \quad \mathrm{E}_1(\Phi) := \{g \in \mathrm{G} : (g\Phi) = \Phi, \det g = 1\} \tag{1.2}$$

form the groups of G–automorphisms and 1–automorphisms of Φ.

The numbers h_{G} and h_1 of G– and 1–equivalence classes for a given discriminant $\Delta \neq 0$ are finite.

For a binary quadratic form $\Phi(x,y)$ over the ring of integers $\mathcal{O}$ of $\mathbf{K}$ a solution of $\Phi(x_0,y_0) = n \in \mathcal{O}$ such that $(x_0,y_0)_{\mathcal{O}} = \mathcal{O}$ is called proper. If $(x_0,y_0)_{\mathcal{O}} = (r)_{\mathcal{O}}$ for some non–unit $r \in \mathcal{O}$, $r \neq 0$, then $\Phi(x_0,y_0)$ is divisible by r^2 and the solution $\Phi(x_0,y_0) = n$ comes from a solution $\Phi(x_1,y_1) = \frac{n}{r^2}$ with $(x_1,y_1)_{\mathcal{O}} = \mathcal{O}$. We define the following *zeta function for proper solutions* for G–equivalence classes of binary quadratic forms over $\mathcal{O}$ of discriminant $\Delta \in \mathcal{O}$, without yet specifying the region of convergence:

1.2. Definition. *Let $\mathrm{N}_{\mathbf{K}\backslash\mathbf{Q}}$ denote the absolute norm. We set*

$$\zeta_\Delta(s) := \sum_{\substack{[\Phi]\\ \Delta(\Phi)=\Delta}} \sum_{\substack{(x,y)\in(\mathcal{O}\times\mathcal{O})/\mathrm{E}(\Phi)\\ (x,y)_{\mathcal{O}}=\mathcal{O}\\ \Phi(x,y)\neq 0}} \frac{1}{|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Phi(x,y))|^s}.$$

$\mathrm{E}(\Phi)$ *denotes the automorphism group of* Φ.

In fact, $\zeta_\Delta(s)$ is independent of the choice of representatives. About the convergence we only remark that if it converges for big enough real part of s then it converges absolutely. So we can change the order of summation. We will see that $\zeta_\Delta(s)$ also makes sense for $\Delta = 0$.

2. A basic identity

One can—as in the classical case over the rational integers—express $\zeta_\Delta(s)$ as a Dirichlet series involving the solutions of quadratic congruences in $\mathcal{O}$. We introduce the following notation.

2.1. Definition. *Fix* $n \in \mathcal{O}$. *We denote by*

$$R_\Phi(n) := \{(x_0, y_0) \in (\mathcal{O} \times \mathcal{O})/\mathrm{E}(\Phi) : (\Phi(x_0, y_0))_\mathcal{O} = (n)_\mathcal{O}, (x_0, y_0)_\mathcal{O} = \mathcal{O}\},$$
$$r_\Phi(n) := |R_\Phi(n)|$$

the set, respectively the cardinality of the set of G*–inequivalent proper representations of the ideal* $(n)_\mathcal{O}$ *by the binary quadratic form* Φ, *and for a set* $\{\Phi_i\}_{i=1,\ldots,h_\Delta}$ *of representatives of* G*–equivalence classes of binary quadratic forms of fixed discriminant* Δ

$$r_\Delta(n) := \sum_{i=1}^{h_\Delta} r_{\Phi_i}(n).$$

We define $k_\Delta(n)$ *to be the cardinality of the following set:*

$$K_\Delta(n) := \{b \in \mathcal{O}/(2n) : \quad b^2 \equiv \Delta \mod (4n)\}.$$

Obviously $k_\Delta(n)$ is finite.

With this definition, $\zeta_\Delta(s) = \sum_{(n)_\mathcal{O} \in \mathcal{O}} \frac{r_\Delta(n)}{|\mathrm{N}_{\mathbf{K}\setminus\mathbf{Q}}(n)|^s}$.

2.2. Lemma. *For all* $(n)_\mathcal{O} \in \mathcal{O}$, $r_\Delta(n) = k_\Delta(n)$.

Proof.

The proof is by a classical idea (cf. [L]).

First we note that for $g, h \in \mathrm{G}$ one has $((gh)\Phi)(x,y) = (g(h\Phi))(x,y)$ and $\mathrm{E}(g\Phi) = g\mathrm{E}(\Phi)g^{-1}$.

Fix $n \in \mathcal{O}$. If $(x_0, y_0)_\mathcal{O} = \mathcal{O}$ and $\Phi(x_0, y_0) = n$ then there exists a matrix $g \in \mathrm{GL}(2, \mathcal{O})$ with first row (x_0, y_0) such that $(g\Phi)(x,y) = nx^2 + b_g xy + c_g y^2$ (here $\det g = 1$). All matrices with the prescribed first row are obtained from g by multiplication from the left with $\begin{pmatrix} 1 & 0 \\ \epsilon\omega & \epsilon \end{pmatrix}$ for $\omega \in \mathcal{O}$ and $\epsilon \in \mathcal{O}^*$. These matrices form a subgroup of G, denoted by B. The resulting matrix we call $g_{\epsilon,\omega}$. With this, $(g_{\epsilon,\omega}\Phi)(x,y) = \epsilon^{-1}nx^2 + b_{\epsilon,\omega}xy + \epsilon c_{\epsilon,\omega}$ and $b_{\epsilon,\omega} = b_{1,0} + 2\omega\epsilon^{-1}n$. Remember that the discriminant remains unchanged: $\Delta =$

$b_{\epsilon,\omega}^2 - 4(n\epsilon^{-1})(\epsilon c_{\epsilon,\omega})$. Furthermore $(\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}\Phi)(x,y) = \epsilon^{-1}nx^2 + bxy + \epsilon cy^2$ if $\Phi(x,y) = nx^2 + bxy + cy^2$.

From this we see that we can choose b modulo $2n$. The variation by $\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}$ (ie. the variation in the determinant) corresponds to the variation in the modulus.

Set $\mathrm{M} := \mathrm{B}\backslash\mathrm{G}/\mathrm{E}(\Phi_i)$. We denote the double coset class of g by $\overline{g}$ Then by the above:

$$\sum_i r_{\Phi_i}(n) = \sum_i \sum_{\substack{\overline{g}\in\mathrm{M} \\ (\Phi((1,0)g)=(n)}} 1 = \sum_i \sum_{\substack{b\in\mathcal{O}/(2n) \\ b^2\equiv\Delta(4n).}} \sum_{\substack{\overline{g}\in\mathrm{M}\text{ s.t. for a }\epsilon\in\mathcal{O}^* \\ (g\Phi)(x,y)=\epsilon^{-1}nx^2+bxy+\epsilon cy^2}} 1.$$

We will show that if such a matrix g exists then it is unique modulo multiplication by a matrix in B from the left and $\mathrm{E}(\Phi)$ from the right. First we fix a form Φ and assume that there exist $g_1, g_2 \in \mathrm{G}$ satisfying $(g_i\Phi)(x,y) = \epsilon_i nx^2 + bxy + \epsilon_i^{-1}cy^2$, $\epsilon_i \in \mathcal{O}^*$, $i = 1,2$. Then $(g_2g_1^{-1})(g_1\Phi)(x,y) = (g_2\Phi)(x,y)$ and $g_2g_1^{-1}$ has the property that it leaves b fixed and multiplies $\epsilon_1 n$ by $\epsilon_2\epsilon_1^{-1}$ and $\epsilon_1^{-1}c$ by $\epsilon_2^{-1}\epsilon_1$. Such a matrix is of the form $\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} T$ where $T \in \mathrm{E}(\Phi)$. For $h := \begin{pmatrix} 1 & 0 \\ 0 & \epsilon_2^{-1}\epsilon_1 \end{pmatrix}$ does the same as $g_2g_1^{-1}$ and $((h^{-1}g_2g_1^{-1})(g_1\Phi)(x,y) = (g_1\Phi)(x,y)$, ie. $h^{-1}g_2g_1^{-1} \in \mathrm{E}(g_1\Phi)$. Hence $g_2g_1^{-1} = hg_1Tg_1^{-1}$ for a $T \in \mathrm{E}(\Phi)$ and equivalently $g_2 = hg_1T$. This was to show.

It follows from the above discussion that

$$\sum_i r_{\Phi_i}(n) = \sum_i \sum_{b\ldots} \delta_{b,i}$$

where $\delta_{b,i}$ is 1 if $[\Phi_i] = [nx^2 + bxy + cy^2]$ and 0 otherwise. Furthermore if $[\Phi_i] \neq [\Phi_j]$ then $b_i \neq b_j$ (by b_i we mean the middle coefficient of the Φ_i after transformation to the leading term n). Hence for given b, $\delta_{b,i} = 1$ exactly once. For $\Delta = 0$ almost all $\delta_{b,i}$ are zero. Thus

$$\sum_i r_{\Phi_i} = \#\{b \in \mathcal{O}/(2n) \text{ s.t. } b^2 \equiv \Delta \mod (4n)\}.$$

□

As a corollary we have:

2.3. Corollary. *For big enough real part of s,*

$$\zeta_\Delta(s) = \sum_{(n)_{\mathcal{O}}\subset\mathcal{O}} \frac{k_\Delta(n)}{\mathrm{N}_{\mathbf{K}\backslash\mathbf{Q}}((n))^s}. \tag{2.3}$$

Since the summation in (2.3) only runs over the principal ideals we cannot yet use the Chinese remainder theorem for further study. Therefore we first define the following zeta function and the corresponding L-series associated to the characters of the ideal class group $\mathcal{I}$ of $\mathcal{O}$:

2.4. Definition. *For a prime ideal $\mathfrak{p} \subset \mathcal{O}$ and $l \geq 0$ ($\mathfrak{p}^0 := \mathcal{O}$) define*

$$k'_\Delta(\mathfrak{p}^l) := |K'_\Delta(\mathfrak{p}, l)| := |\{b \in \mathcal{O}/\mathfrak{p}^l : \quad b^2 \equiv \Delta \mod \mathfrak{p}^l\}|.$$

For a character χ of $\mathcal{I}$ and $\Re s > 1$ one can define the local factors

$$Z_\Delta(\chi, \mathfrak{p}, s) := \begin{cases} \sum_{l=0}^\infty \frac{k'_\Delta(\mathfrak{p}^l)\chi(\mathfrak{p}^l)}{\mathrm{N}(\mathfrak{p}^l)^s}, & \text{for } \mathfrak{p} \nmid 2, \\ \sum_{l=0}^\infty \frac{k'_\Delta(\mathfrak{p}^{l+2e})\chi(\mathfrak{p}^l)}{\mathrm{N}(\mathfrak{p}^l)^s}, & \text{for } \mathfrak{p}^e \| 2, \end{cases}$$

and

$$Z_\Delta(\chi, s) := \prod_{\substack{\mathfrak{p} \subset \mathcal{O} \\ \text{prime}}} Z_\Delta(\chi, \mathfrak{p}, s).$$

2.5. Lemma. *Let n denote the degree of $\mathbf{K}$ over $\mathbf{Q}$ and h the class number of $\mathbf{K}$. Then*

$$\zeta_\Delta(s) = \frac{1}{2^n h} \sum_\chi Z_\Delta(\chi, s), \tag{7.4}$$

where the sum runs over all characters of $\mathcal{I}$.

Proof. By construction together with application of the inversion formula given in [EGM, Lemma 3.6] the sum in the right hand side of formula (7.4) is equal to

$$\sum_{(n) \subset \mathcal{O}} \frac{k'_\Delta(4n)}{\mathrm{N}((n))^s}.$$

It remains to show $k'_\Delta(4n) = 2^n k_\Delta(n)$. If b is in $K_\Delta(n)$ then $b_\epsilon := b + 2\epsilon n \in K'_\Delta(4n)$ for each unit $\epsilon \in \mathcal{O}^*$. b_ϵ and b_η are equivalent modulo $(4n)$ if and only if $\epsilon \equiv \eta \mod (2)$. Hence $b \in K_\Delta(n)$ gives $\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(2) = 2^n$ different elements in $K'_\Delta(4n)$. Vice versa for each $b \in K'_\Delta(4n)$ one can find an $\epsilon \in \mathcal{O}^* \cup \{0\}$ such that b_ϵ is already given modulo $(2n)$. The assertion follows. $\square$

3. Computation of the local factors

One computes $k'_\Delta(\mathfrak{p}^l)$ by localization. For $\mathfrak{p}$ be a prime ideal in $\mathcal{O}$ lying over the prime $p \in \mathbf{Z}$ one has the isomorphism $\mathcal{O}/\mathfrak{p} \simeq \mathbf{F}_{p^f}$, the finite field of p^f elements, where f denotes the residue degree of $\mathfrak{p}$ over p. $\mathcal{O}/\mathfrak{p}^l$ is isomorphic to the ring of Witt vectors $W_l(\mathbf{F}_{p^f})$ of lenght l over F_{p^f}. Each $a \in \mathcal{O}$ is congruent to the sum $a \equiv a_0 + a_1\pi + \cdots + a_{l-1}\pi^{l-1} \mod \mathfrak{p}^l$ for some $\pi \in \mathfrak{p}$ such that $\mathfrak{p} \| \pi$ and $a_i \in \mathcal{O}/\mathfrak{p}$ for all i. Furthermore $\mathcal{O}/\mathfrak{p}^l \simeq \hat{\mathcal{O}}_\mathfrak{p}/\pi^l\hat{\mathcal{O}}_\mathfrak{p}$, where $\hat{\mathcal{O}}_\mathfrak{p}$ denotes the completion of $\mathcal{O}_\mathfrak{p}$. So one can work with principal ideals and $k'_\Delta(\mathfrak{p}^l) = k'_d(\pi^l)$ where d corresponds to Δ in the localization and $k'_d(\pi^l)$ is defined the obvious way.

One has to look at different cases seperately. We only give the results. The proof uses the pigeon hole principle. First we consider the case that $\mathfrak{p}$ does not divide Δ. The first lemma is a generalization of [L, Satz 87, Satz 97].

3.1. Lemma. *For $\Delta \in \mathcal{O}$ and a prime ideal $\mathfrak{p} \subset \mathcal{O}$ such that $\mathfrak{p} \nmid \Delta$ the number $k'_\Delta(\mathfrak{p}^l)$ is equal to*

$$k'_\Delta(\mathfrak{p}^l) = \begin{cases} 1, & \text{for } l = 0, \\ 2^{f[\frac{l}{2}]}, & \text{for } \mathfrak{p}^e \| 2, \quad 1 \le l \le 2e, \text{ if } \Delta \text{ is a square} \mod \mathfrak{p}^l, \\ 2^{fe+1}, & \text{for } \mathfrak{p}^e \| 2, \quad 2e+1 \le l, \text{ if } \Delta \text{ is a square} \mod \mathfrak{p}^{2e+1}, \\ 0, & \text{for } \mathfrak{p}^e \| 2, \quad 1 \le l, \text{ if } \Delta \text{ is not a square} \mod \mathfrak{p}^{\min(l,2e+1)}, \\ 1 + \left(\frac{\Delta}{\mathfrak{p}}\right), & \text{for } \mathfrak{p} \nmid 2, \quad 1 \le l. \end{cases}$$

with $2^f = N_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p})$ for $\mathfrak{p}|2$, $[\frac{l}{2}]$ the Gauss bracket of $\frac{l}{2}$, and the generalized Legendre symbol $\left(\frac{\Delta}{\mathfrak{p}}\right) = \begin{cases} 1, & \text{if } \Delta \text{ is a square} \mod \mathfrak{p}, \\ -1, & \text{if not,} \end{cases}$.

The argumentation is as follows: One determines

i) the number Q_{D_l} of elements in the set of definition which fulfil the divisibility condition imposed by Δ on the squareroot, here: the number of elements in $\mathcal{O}/\mathfrak{p}^l$ that are not divisible by $\mathfrak{p}$;

ii) the maximal number $Q_{\max_l}$ of possible solutions of the congruence for fixed Δ

iii) conditions 'E_l' which arise from squaring a number of the set in i) and the number Q_{E_l} of elements in $\mathcal{O}/\mathfrak{p}^l$ that fulfil 'E_l'. By the pigeon hole principle, if $Q_{\max_l} = \frac{Q_{D_l}}{Q_{E_l}}$ then $Q_{\max_l}$ is the exact number of solutions. This is equivalent to the statement that the conditions 'E_l' are necessary and sufficient. For primes dividing 2 the one doesn't have the convenience of applicability of Hensel's lemma. This case is more tedious than in the situation over $\mathbf{Q}$, whereas the other case in this lemma is completely analoguous to the classical case.

Before we turn to the case that $\mathfrak{p}$ divides Δ we have to do some preparation.

3.2. Definition. *Fix a prime ideal $\mathfrak{p} \in \mathcal{O}$ and $(\pi)_{\mathcal{O}_\mathfrak{p}} = \mathfrak{p}\mathcal{O}_\mathfrak{p}$. Given $a \in \mathcal{O}_\mathfrak{p}^*$ define*

$$k_a^\pi(l,m) := |\{b \in \mathcal{O}_\mathfrak{p}/\pi^l : \quad b^2 \equiv a\pi^m \mod \pi^l\}|$$

for $m, l \ge 0$.

Let $N(\mathfrak{p})$ denote the absolute norm of $\mathfrak{p}$.

3.3. Lemma. *Let $\mathfrak{p} \in \mathcal{O}$ be prime ideal and $(\pi)_{\mathcal{O}_\mathfrak{p}} = \mathfrak{p}\mathcal{O}_\mathfrak{p}$, $a \in \mathcal{O}_\mathfrak{p}^*$. Then*

$$k_a^\pi(l,m) = \begin{cases} N(\mathfrak{p})^{[\frac{l}{2}]}, & \text{for } m \ge l, \\ 0, & \text{for } m < l, \ m \text{ odd}, \\ N(\mathfrak{p})^{\frac{m}{2}} k_a^\pi(l-m,0), & \text{for } m < l, \ m \text{ even}. \end{cases}$$

Proof. First one observes that $k_a^\pi(l,m) = N(\mathfrak{p})k_a^\pi(l-2,m-2)$. If n is a solution of $x^2 \equiv a\pi^{m-2} \mod \pi^{l-2}$ then $\pi(n + \pi^{l-2}r)$, $r = 0, 1, \ldots, \pi - 1$, are the solutions of the congruence with respect to l and m. By repeating the reduction the last case follows.

For $m \geq l$ the congruence reduces to $x^2 \equiv 0 \mod \pi^l$. All multiples of $\pi^{[\frac{l+1}{2}]}$, of which there are $N(\mathfrak{p})^{[\frac{l}{2}]}$ different ones in $\hat{\mathcal{O}}_\mathfrak{p}/\pi^l\hat{\mathcal{O}}_\mathfrak{p}$, solve this.

If m is odd and smaller than l it follows from the congruence condition that $\pi|(a+r\pi^{l-m})$ for some r. Since a was supposed not to be divisible by π one gets a contradiction to the maximality of m. $\square$

Next we count the number of solutions b in $\mathcal{O}/\mathfrak{p}^l$ of the quadratic congruence $b^2 \equiv \Delta \mod \mathfrak{p}^l$ if $\mathfrak{p}$ divides Δ.

3.4. Definition. *Take $\Delta \neq 0$ and let $m \in \mathbb{N}$ be such that $\mathfrak{p}^m \| \Delta$. Let first $\mathfrak{p}$ be a prime ideal in $\mathcal{O}$ not dividing 2. Let $d = \pi^m a$ correspond to Δ in the localization of $\mathcal{O}$ by $\mathfrak{p}$ with uniformizing element π. Set*

$$\left(\frac{\Delta, m}{\mathfrak{p}}\right) := \begin{cases} 1, & \text{if } a \text{ is a square} \mod \pi \\ -1, & \text{if not.} \end{cases}$$

$$\left(\frac{\Delta}{\mathfrak{p}}\right) := \begin{cases} 0, & \text{if } \mathfrak{p}|\Delta \\ 1, & \text{if } \mathfrak{p} \nmid \Delta \text{ and } \Delta \text{ is a square} \mod \mathfrak{p} \\ -1, & \text{if } \mathfrak{p} \nmid \Delta \text{ and } \Delta \text{ is not a square} \mod \mathfrak{p}. \end{cases}$$

For $\mathfrak{p}$ and $e \in \mathbb{N}$ such that $\mathfrak{p}^e \| 2$ define

$$\left(\frac{\Delta, m}{\mathfrak{p}}\right) := \begin{cases} 1, & \text{if } a \text{ is a square} \mod \pi^{2e+1} \\ -1, & \text{if not,} \end{cases}$$

$$\left(\frac{\Delta}{\mathfrak{p}}\right) := \begin{cases} 0, & \text{if } \mathfrak{p}|\Delta \\ 1, & \text{if } \mathfrak{p} \nmid \Delta \text{ and } \Delta \text{ is a square} \mod \mathfrak{p}^{2e+1} \\ -1, & \text{if } \mathfrak{p} \nmid \Delta \text{ and } \Delta \text{ is not a square} \mod \mathfrak{p}^{2e+1}. \end{cases}$$

These are generalized Legendre symbols.

Now we can formulate the results for $\mathfrak{p}|\Delta$.

3.5. Lemma. *Fix $\Delta \in \mathcal{O}$. Let $\mathfrak{p} \in \mathcal{O}$ a prime ideal that does not divide 2. For $\Delta \neq 0$ let $m \in \mathbb{N}$ be such that $\mathfrak{p}^m \| \Delta$. Then the number $k'_\Delta(\mathfrak{p}^l)$ of $x \in \mathcal{O}/(\mathfrak{p}^l)$ such that $x^2 \equiv \Delta \mod \mathfrak{p}^l$ is equal to*

$$k'_\Delta(\mathfrak{p}^l) = \begin{cases} N(\mathfrak{p})^{[\frac{l}{2}]}, & \text{for } 0 \leq l \leq m, \text{ or } \Delta = 0 \text{ and } l \geq 0 \\ 0, & \text{for } \Delta \neq 0,\ m \text{ odd},\ l > m \\ \left(1 + \left(\frac{\Delta, m}{\mathfrak{p}}\right)\right) N(\mathfrak{p})^{\frac{m}{2}}, & \text{for } \Delta \neq 0,\ m \text{ even},\ l > m\ . \end{cases}$$

Let χ be a character of the ideal class group $\mathcal{I}$ of $\mathcal{O}$. The generating series $Q^\mathfrak{p}_\Delta(x,\chi) = \sum_{l=0}^\infty \chi(\mathfrak{p}^l)k'_\Delta(\mathfrak{p}^l)x^l$ for $\chi(\mathfrak{p}^l)k'_\Delta(\mathfrak{p}^l)$ with respect to l is equal to

$$Q^\mathfrak{p}_\Delta(x,\chi) = \begin{cases} \frac{1-(\chi(\mathfrak{p})x)^2}{(1-(\chi(\mathfrak{p})x))(1-N(\mathfrak{p})(\chi(\mathfrak{p})x)^2)}, & \text{for } \Delta = 0 \\ \frac{(1-(\chi(\mathfrak{p})x)^2)(1-(N(\mathfrak{p})(\chi(\mathfrak{p})x)^2)^{n+1})}{(1-(\chi(\mathfrak{p})x))(1-N(\mathfrak{p})(\chi(\mathfrak{p})x)^2)}, & \text{for } \Delta \neq 0,\ m = 2n+1 \\ \frac{(1-(N(\mathfrak{p})(\chi(\mathfrak{p})x)^2)^n)}{(1-N(\mathfrak{p})(\chi(\mathfrak{p})x)^2)}(1 + (\chi(\mathfrak{p})x)) + & \\ \quad + \frac{(1+(\frac{\Delta,m}{\mathfrak{p}})(\chi(\mathfrak{p})x))}{1-(\chi(\mathfrak{p})x)} N(\mathfrak{p})^n(\chi(\mathfrak{p})x)^{2n}, & \text{for } \Delta \neq 0,\ m = 2n. \end{cases}$$

Proof. Take an uniformizing element π for $\mathfrak{p}$, let d be the image of Δ in the localization with respect to $\mathfrak{p}$ and $d = \pi^m a$ for appropriate $a \in \mathcal{O}_{\mathfrak{p}}$. Then $k'_\Delta(\mathfrak{p}^l) = k^\pi_a(l,m)$. At this point we can apply Lemma 3.3. For even m and $m < l$ we get $k^\pi_a(l-m,0)$ by Lemma 3.1. The formula for the generating series follows easily. □

For prime ideals which divide 2 we determine the generating series with a shift of l by $2\times$ the ramification degree over 2 in view of Definition 2.4 and Lemma 2.5.

3.6. Lemma. *Fix $\Delta \in \mathcal{O}$. Let $\mathfrak{p} \in \mathcal{O}$ a prime ideal that divides 2 with ramification degree e. For $\Delta \neq 0$ let $m \in \mathbf{N}$ be such that $\mathfrak{p}^m \| \Delta$. Then the number $k'_\Delta(\mathfrak{p}^l)$ of $x \in \mathcal{O}/(\mathfrak{p}^l)$ such that $x^2 \equiv \Delta \mod \mathfrak{p}^l$ is equal to*

$$k'_\Delta(\mathfrak{p}^l) = \begin{cases} N(\mathfrak{p})^{[\frac{l}{2}]}, & \text{for } 0 \leq l \leq m, \text{ or } \Delta = 0 \text{ and } l \geq 0 \\ 0, & \text{for } \Delta \neq 0,\ m \text{ odd},\ l > m \\ N(\mathfrak{p})^{[\frac{l}{2}]}, & \text{for } \Delta \neq 0 \text{ a square } \mod \mathfrak{p}^{l-m}, \\ & m \text{ even},\ m < l \leq 2e+m \\ \left(1 + \left(\frac{\Delta, m}{\mathfrak{p}}\right)\right) N(\mathfrak{p})^{\frac{m}{2}+e}, & \text{for } \Delta \neq 0,\ m \text{ even},\ 2e+m < l\,. \end{cases}$$

The generating function for $\chi(\mathfrak{p}^l)k'_\Delta(\mathfrak{p}^{l+2e})$ with respect to l is equal to

$$Q^{\mathfrak{p}}_\Delta(x,\chi) = \begin{cases} \mathrm{N}(\mathfrak{p})^e \frac{1-(\chi(\mathfrak{p})x)^2}{(1-(\chi(\mathfrak{p})x))(1-N(\mathfrak{p})(\chi(\mathfrak{p})x)^2)}, & \text{for } \Delta = 0 \\ \mathrm{N}(\mathfrak{p})^e \frac{(1-(\chi(\mathfrak{p})x)^2)(1-(N(\mathfrak{p})(\chi(\mathfrak{p})x)^2)^{n+1})}{(1-(\chi(\mathfrak{p})x))(1-N(\mathfrak{p})(\chi(\mathfrak{p})x)^2)}, & \text{for } \Delta \neq 0,\ m = 2n+1 \\ \mathrm{N}(\mathfrak{p})^e \left(\frac{(1-(N(\mathfrak{p})(\chi(\mathfrak{p})x)^2)^n)}{(1-N(\mathfrak{p})(\chi(\mathfrak{p})x)^2)}(1+(\chi(\mathfrak{p})x)) + \right. & \\ \left. + \frac{(1+(\frac{\Delta, m}{\mathfrak{p}}))(\chi(\mathfrak{p})x))}{1-(\chi(\mathfrak{p})x)} N(\mathfrak{p})^n (\chi(\mathfrak{p})x)^{2n}\right), & \text{for } \Delta \neq 0,\ m = 2n, \\ 0, & \text{if } \Delta \text{ not a square } \mod \mathfrak{p}^l \\ & \text{for some } 0 < l \leq 2e. \end{cases}$$

Proof. The same way as Lemma 3.5.

Over $\mathbf{Z}$ such formulas can be derived from [Hi–Z, cpt 1.2, (23)]. See also [Z2, Par. 4. Prop. 3 iii.].

4. Results

Specialization of the generating series at $x = \mathrm{N}(\mathfrak{p})^s$ and comparison of Euler factors gives the desired expression for Z_Δ.

4.1. Theorem. *For $\Delta \in \mathcal{O}$, χ a character of the ideal class group of $\mathbf{K}$, $\mathfrak{p}$ a prime ideal in $\mathcal{O}$ and a complex number s with big enough real part, the following identities hold*

$$Z_\Delta(\chi, \mathfrak{p}, s) = Q^{\mathfrak{p}}_\Delta(\mathrm{N}(\mathfrak{p})^s) \tag{4.1}$$

and

(4.2)

$$Z_\Delta(\chi,s)=\begin{cases} 2^{[\mathbf{K}:\mathbf{Q}]}\frac{L_\chi(s)L_\chi(2s-1)}{L_\chi(2s)}, & \textit{if } \Delta=0,\\ 0, & \Delta\neq 0 \textit{not a square} \mod \mathfrak{p}^l \\ & \textit{for a prime ideal } \mathfrak{p}|2 \textit{ and } 0<l\leq 2e_{\mathfrak{p}},\\ 2^{[\mathbf{K}:\mathbf{Q}]}\frac{L_\chi(s)}{L_\chi(2s)}L_{(\frac{\Delta}{\cdot}),\chi}(s)z_{\Delta,\chi}(s), & \textit{otherwise,}\end{cases}$$

where $L_{(\frac{\Delta}{\cdot}),\chi}(s)=\prod_{\substack{\mathfrak{p}\subset\mathcal{O}\\ \text{prime}}}\frac{1}{1-(\frac{\Delta}{\mathfrak{p}})\chi(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-s}}$, $z_{\Delta,\chi}(s)=\prod_{\mathfrak{p}|\Delta \text{ prime}}\frac{Q_\Delta^{\mathfrak{p}}(\mathrm{N}(\mathfrak{p})^{-s},\chi)}{1+\chi(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-s}}$ *and* $L_\chi(s)$ *is the L-series for* $\mathbf{K}$ *associated to the character* χ *of the ideal class group* $\mathcal{I}$. *Furthermore it follows that* $Z_\Delta(\chi,s)$ *is meromorphic in* $\mathbf{C}$.

If Δ is a square, then $L_{(\frac{\Delta}{\cdot}),\chi}(s)=L_\chi(s)$.

Over $\mathbf{Q}$ and imaginary quadratic number fields one can also consider 1–equivalence of binary quadratic forms. The zeta function defined in an analogous way to Definition 1.3 but for 1–equivalence. For arbitrary $\Delta\in\mathcal{O}$ we can show the following relation between $\zeta_\Delta^1(s)$ and ζ_Δ.

4.2. Lemma.

$$\zeta_\Delta^1(s)=|\mathcal{O}^*|\zeta_\Delta(s)=\frac{|\mathcal{O}^*|}{2^{[\mathbf{K}:\mathbf{Q}]}h}\sum_\chi Z_\Delta(\chi,s).$$

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