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Algebraic Models of Organizations

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1. Introduction

H.A.Simon[3] has suggested a framework that permits a comparison of the economist's theory of the firm and the theory of organizational equilibrium. In his paper, the former theory is defined as "F-theory" and the latter as "O-theory". He obtained a result that the F-theory solution, in the case of perfect competetion, is identical with the particular O-theory solution that is optimal to the entrepreneur.

In this paper we will induce a corresponding result in an algebraic framework on contrast to Simon's analytical framework. Our Framework is more general in the meanings that Simon's is one of special cases of ours.

2. Models of Participants

Definition 1　　Behavior Selection Model

A five-tuple (U,Y,M,g,Ψ) is called a "behavior selection model", if the following conditions hold.

1) U, Y, and M are sets. They are called an inducement set, a contribution set and an altenative set, respectively. $M \subset U \times Y$ is

supposed.

2) g is a funcition from M to the set of real numbers. It is called an objective funciton. The pair (M,g) is called decision problem.

3) Ψ is a symbol of a particular operator which defines a subset $\Psi(M,g) \subset M$ for a given decision problem (M,g). Ψ is called a desicion principle. Details will be defined in Definition 3.

$$(M,g) \xrightarrow{\Psi} \Psi(M,g) \subset M$$

Figure 1 decision princile

In this paper, the inducement set U and the contribution set Y are fixed. So a behavior selection model will be denoted briefly by $S = \Psi(M,g)$.

Assumption 1

1) Objective function

The reverse funciton $g^{-1}(r): U \rightarrow Y$ are "one to one" and "onto" for all real number r.

2) Alternative sets with parameter p

There exists a class of functions $\{M(p): U \rightarrow Y\}$ such that $M = \cup M(p)$ and $(\forall p)(\forall q)(\ p \neq q \rightarrow M(p) \cap M(q) = \phi\ )$.

Definition 2 Operators (Sat, Max)

For any decision problem (M,g), we can define the following two sbsets of M.

$m \varepsilon \mathrm{Sat}(M,g) \leftarrow\rightarrow g(m) \geqq 0,$

$m \varepsilon \mathrm{Max}(M,g) \leftarrow\rightarrow (\forall m')(\ g(m) \geqq g(m')\ ).$

Definition 3 Decision Principles

1) If $\Psi(M,g)=Sat(M,g)$, Ψ is called a "satisfactory principle".

2) If $\Psi(M,g)=Max(M,g)$, Ψ is called an "optimazing principle".

3) If $\Psi(M,g)=\bigcup Max(M(p),g)$, Ψ is called an "optimizing principle with parameter p ".

Lemma 1 Properties of Operators

For any decision problem (M,g), the followings hold.

1) $Sat(M,g)=\bigcup Sat(M(p),g)$, where $\bigcup$ is a union in terms of parameters p.

2) If $Sat(M,g)\neq\phi$, then $Max(Sat(M,g),g)=Max(M,g)$ where ϕ is a symbol of empty set.

3. Complete Competetion

Definition 4 (q,r)-completeness

1) If there exists an parameter q such that

$$Max(M(p),g)=\begin{cases} M(q) & \text{if } p=q \\ \phi & \text{if } p\neq q \end{cases}$$

holds, an objective function g is called "complete".

2) If g is complete, then there exists an real number r such that $M(q)=g^{-1}(r)$. Hence the objective function g can be called "(q,r) -complete".

Definition 5 Restricted Set

For any set $S\subset U\times Y$, a "restricted set" $S[g(m)=r]\subset S$ is defined as follows.

$$S[g(m)=r]=\{m\in S\mid g(m)=r\}.$$

Lemma 2 Property of Completeness

Let an objective funciton g be (q,r)-complete and $r \geqq 0$.

Then ,

$$S(p)[g(m)=r]=\begin{cases} M(q) & \text{if } p=q \\ \phi & \text{if } p \neq q \end{cases} .$$

holds for $S(p)=Sat(M(p),g)$.

Proposition 3

Let an objective funcition g be (q,r)-complete and $r \geqq 0$.

And let $S=Sat(M,g)$ and $S\hat{}=\bigcup Max(M(p),g)$. Then $S[g(m)=r]=S\hat{}=M(q)$.

4. Models of Organizations

Definition 6 Organization (R,{Si})

An "organization with n participants" is defined by a pair (R,{Si}) such that

$$Si=\Psi i(Mi,gi) \quad (i=1,2,\ldots,n)$$

$$U=U1\times\cdots\times Un$$

$$Y=Y1\times\cdots\times Yn$$

$$R\subset U\times Y$$

where Si, R and $S= R\cap(S1\times S2\times\cdots\times Sn)$ is called a "participant", an "organizational restriction" and an "organizational behavior", respectively.

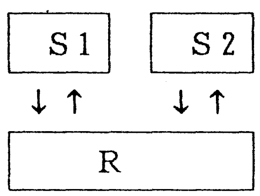

Figure 2 Structure of an organization

Assumption 2

1) We assume $proj1(R)=M1$ and $proj2(R)=M2$ where $proj1: U \times Y \rightarrow U1 \times Y1$ is a projection function.

2) From now on, we will focus on the case $n=2$, i.e., two-participants organizations.

Definition 7 Internal Model

If the alternative set of participant 1 satisfies

$$M1=\{m1 \mid \exists m2 \in S2, (m1,m2) \in R\},$$

then parcipant 1 is called to "have an internal model of participant 2".

Proposition 4

Let $(R,\{S1, S2\})$ be an organization.

If participant 1 have a internal model of participant 2, then

$$S \neq \phi \leftrightarrow S1 \neq \phi .$$

5. Comparison of Organizations

Definition 8 Optimum Set of Organization

For any organizational behavior S of any organization $(R, \{S1, S2\})$, the following set

$$OPT1(S)=Max(proj1(S),g1)$$

is called an "optimum set" of the organization $(R,\{S1, S2\})$.

Proposition 5

Assume that $\Psi 1=Sat$ in an organization $(R,\{S1,S2\})$.

If $S \neq \phi$, then

$$OPT1(R \cap (S1 \times S2))=OPT1(R \cap (M1 \times S2)) .$$

Let us consider two organizations such that organizational restriction R and objective functions gi are common. It means that decision principles Ψi and alternative sets Mi might be differnt. The optimum sets of those organizations below will be compared. Those organizations correspond to "O-theory" and "F-theory", respectively.

Organization 1

$S1=Sat(M1,g1)$, $S2=Sat(M2,g2)$, $S=R\cap(S1\times S2)$.

Organization 2

$S\hat{}1=Max(M1\hat{},g1)$, $S\hat{}2=\bigcup Max(M2(p),g2)$, $S\hat{}=R\cap(S\hat{}1\times S\hat{}2)$.

where $M\hat{}1 \;=\{m1 \mid \exists m2 \in S2,\ (m1,m2)\in R\}$

(i.e. internal model of participant 2)

$\bigcup M2(p)=M2$

Theorem 6

Assume that the common objective function g2 is (q,r)-complete in the above two organizations. If $S[g2(m2)=r]\neq\phi$, then $OPT1(S[g2(m2)=r])=S1\hat{}$.

6. Discussion

Theorem 6 is the main result of this paper. The set S[g2(m2)=r] in the left hand side is an organizational behavior under the condition that the utility value of the participant 2 is restricted to a fixed r. Then, the optimum set

OPT1(S[g2(m2)=r])

of Organization 1 corresponds to the optimum solution to entrepreneur in "O-theory". On the other hand, the participant 1 of Organization 2 have an internal model of participant 2. Then S1^ corresponds to the optimum solution of "F-theory" in the case of complete competetion. Theorem 6 shows that both are identical.

As mentioned in Introduction, our framework is algebraic and including Simon's framework as a special case. Indeed, we can get the complete competition condition ($\phi(u)=qu$) with additional properties. Let us assume that $U = Y =$ (the set of positive real numbers) and that $M(p) = \{ (u,y) \mid y = pu \}$. Then $M=\bigcup M(p)=U\times Y$. And let us define $g: M\rightarrow$ (real numbers) by

$$g(u,y) = \phi(u) - y$$

such that $\phi(u)$ is an utility function of inducement u and that $\phi(u)$ is differntiable. If $g(0,0) = 0$, then we have $\phi(u)=qu$ from (q,r)-completeness of g. Therfore S1^ of Organization 2 is the optimum solution of the entrepreneur.

References

[1] P.A.Samuelson, Economics (7th.ed.), McGraw-hill (1980)

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[3] H.A.Simon, Models of Man , John Wiley & Sons (1957)

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Appendices

Proof of Lemma 1

1) Since

$$m \in \bigcup Sat(M(p),g)$$
$$\leftrightarrow \exists p\ ,\ m \in Sat(M(p),g)$$
$$\leftrightarrow \exists p\ ,\ m \in M(p),\ g(m) \geqq 0$$
$$\leftrightarrow m \in \bigcup M(p)=M,\ g(m) \geqq 0$$
$$\leftrightarrow m \in Sat(M,g)$$

hold, we have $\bigcup Sat(M(p),g)=Sat(M,g)$.

2) Since $Sat(M,g) \neq \phi$, $\exists m^* \in M,\ g(m^*) \geqq 0$. Then

$$m \in Max(M,g) \rightarrow g(m) \geqq g(m^*) \geqq 0 \rightarrow m \in Sat(M,g).$$

That is, $Max(M,g) \subset Sat(M,g)$. Then we have $Max(Sat(M,g),g) = Sat(M,g)$. ■

Proof of Lemma 2

Let an objective function g be (q,r)-complete. From Definition 4, we have $M(q)=g^{-1}(r)$. Then the condition $r \geqq 0$ implies $S(p)[g(m)=r] = M(p) \cap M(q)$. Indeed,

$$m \in S(p)[g(m)=r]$$
$$\leftrightarrow m \in S(p)=Sat(M(p),g),\ g(m)=r$$
$$\leftrightarrow m \in M(p),\ g(m)=r \geqq 0$$
$$\leftrightarrow m \in M(p) \cap g^{-1}(r)=M(p) \cap M(q)\ .$$

Hence Assumption 1 (2) implies the result which is to be proved.

■

Proof of Proposition 3

Firstly, let an objective function g be (q,r)-complete. From Definition 4, we have

$$Max(M(p),g)=\begin{cases} M(q) & \text{if } p=q \\ \phi & \text{if } p\neq q \end{cases}$$

Then $S\hat{}=\cup Max(M(p),g)=M(q)$.

Secondly, let $S(p)=Sat(M(p),g)$. Lemma 1 (1) implies

$$S=Sat(M,g)=\cup Sat(M(p),g)=\cup S(p).$$

Then we will have $S[g(m)=r]=\cup(S(p)[g(m)=r])$. Indeed,

$m \in S[g(m)=r]$

$\leftrightarrow$ $m \in S$, $g(m)=r$

$\leftrightarrow$ $\exists p$, $m \in S(p)$, $g(m)=r$

$\leftrightarrow$ $\exists p$, $m \in S(p)[g(m)=r]$

$\leftrightarrow$ $m \in \cup S(p)[g(m)=r]$.

Hence Lmma 2 implies $S[g(m)=r]=M(q)$. ■

Proof of Proposition 4

(nessesity)

It is trivial since $S=R\cap(S1\times S2)\neq\phi \rightarrow S1\neq\phi$.

(sufficiency)

$m1 \in S1$

$\rightarrow$ $m1 \in M1=\{m1 \mid \exists m2 \in S2, (m1,m2) \in R\}$

$\rightarrow$ $\exists m2 \in S2, (m1,m2) \in R$

$\rightarrow$ $(m1,m2) \in R\cap(S1\times S2)=S$. ■

Proof of Proposition 5

Let $S=R\cap(S1\times S2)$ and $S*=R\cap(M1\times S2)$.

Then, $proj1(S)=Sat(proj1(S*),g1)$. Indeed,

$m1\,\varepsilon\, proj1(S)$

$\leftarrow\rightarrow\ \exists m2,\ (m1,m2)\,\varepsilon\, S$

$\leftarrow\rightarrow\ \exists m2\,\varepsilon\, S2,\ (m1,m2)\,\varepsilon\, R, m1\,\varepsilon\, S1$

$\leftarrow\rightarrow\ m1\,\varepsilon\, proj1(S*), g1(m1)\geqq 0$

$\leftarrow\rightarrow\ m1\,\varepsilon\, Sat(proj1(S*),g1)$.

Since $S\neq\phi$ from assumption, we have

$Sat(proj1(S*),g1)=proj1(S)\neq\phi$. Therefore Lemma 1 (2) implies

$OPT1(S) = Max(\ proj1(S),g1) = Max(\ Sat(proj1(S*),g1),\ g1\ )$

$= Max(\ proj1(S*),\ g1\ ) = OPT1(S*)$. ■

Proof of Theorem 6

Firstly, from the definition of organizational restriction, we have

$$S[g2(m2)=r]=R\cap(S1\times(S2[g2(m2)=r])\ ) .$$

And from assumption $S[g2(m2)=r]\neq\phi$, we have $S2[g2(m2)=r])\neq\phi$.

Then $r=g2(m2)\geqq 0$.

Secondly, Proposition 5 implies

$$OPT1(S[g2(m2)=r])=OPT1(R\cap(M1\times(S2[g2(m2)=r])\ ).$$

Thirdly, from the definition of participant 2 in Organization 2, we have

$$S1\hat{}=OPT1(\ R\cap(M1\times S2\hat{})\ ) .$$

On the other hand, since $r\geqq 0$ and g2 is (q,r)-complete, Proposition 3 implies

$$S2\hat{}=S2[g2(m2)=r] .$$

Therefore $OPT1(S[g2(m2)=r])=S1\hat{}$. ■