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Infinitary Jónsson functions and elementary embeddings

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Abstract. We apply Erdös-Hajnal's partition theorem to combinatorics on $[\lambda]^\lambda$ and derive Kunen's theorem on nontrivial elementary embeddings as a corollary. Pushing this idea further, we also show that a positive answer to Magidor's question improves Kunen's theorem.

0. INTRODUCTION

Elementary embeddings between transitive models of ZFC have been central notions in set theory. Requiring the target model of the embedding to be larger, we get stronger properties and closer to inconsistency.

The following limitation for the size of the target model is highly celebrated.

THEOREM 1 (KUNEN [6]). *Let $j : V \to M$ be a nontrivial elementary embedding and λ the least fixed point of j above the critical point of j. Then $j''\lambda \notin M$.*

Various proofs have been provided for Theorem 1 [1, 4, 5], but Kunen's original proof is still of some interest because of its simplicity. In this note, we take an alternative approach to the proof of Theorem 1 via Erdös-Hajnal's partition theorem [2] used also by Kunen, working with the canonically induced ultrafilter on $[\lambda]^\lambda$ rather than the embedding $j : V \to M$ with $j''\lambda \in M$. The ultrafilter is strongly normal and nonprincipal as is well known to be for supercompact or huge embeddings [8]. But an ω-Jónsson function on λ (i.e. a witness for $\lambda \not\to [\lambda]^\omega_\lambda$) gives us the nonexistence of such a filter and hence the desired contradiction.

Considering a bounded ω-Jónsson function on λ (i.e. a witness for $\lambda \not\to [\lambda]^{\omega,b}_\lambda$), whose existence was asked by Magidor [5], a strongly seminormal filter and an extender (i.e. a direct system of ultrafilters) instead of an ω-Jónsson function, a strongly normal filter and a single ultrafilter respectively, we obtain our main result.

THEOREM 2. *Let $j : V \to M$ and λ be as in Theorem 1. Assume that a bounded ω-Jónsson function on λ exists. Then $j''\alpha \notin M$ for some $\alpha < \lambda$.*

Therefore Theorem 1 will be improved if Magidor's question is solved positively for strong limit cardinals of countable cofinality (in fact, a slightly weaker condition suffices).

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1. Preliminaries

First we introduce some notions and notations. We use κ, λ and μ to stand for an infinite ordinal. By $(x)^\mu$, we denote the set $\{y \subset x : \text{ot } y = \mu\}$, where x is a set of ordinals and ot y is the order type of $(y, \in)$. By $[x]^\mu$ (resp. $[x]^\mu_b$), we denote the set $\{y \subset x : |y| = \mu\}$ (resp. $\{y \in [x]^\mu : \sup y < \sup x\}$), when μ is an infinite cardinal.

F is said to be a filter on $[\lambda]^\mu$ if $F \subsetneq P[\lambda]^\mu$ is closed under intersection and superset and is fine, i.e. $\{x \in [\lambda]^\mu : x \ni \alpha\} \in F$ for any $\alpha < \lambda$. A filter F on $[\lambda]^\mu$ is said to be κ-complete if $\bigcap X \in F$ for any $X \subset F$ with $|X| < \kappa$, principal if $\bigcap F \neq 0$, and an ultrafilter if F is maximal with respect to inclusion. Also, F is said to be strongly normal (resp. strongly seminormal) if $\Delta_{a \in [\lambda]^\omega} X_a = \{x \in [\lambda]^\mu : \forall a \in [x]^\omega \; x \in X_a\} \in F$ (resp. $\Delta_{a \in [\lambda]^\omega_b} X_a = \{x \in [\lambda]^\mu : \forall a \in [x]^\omega_b \; x \in X_a\} \in F$) for any $\{X_a : a \in [\lambda]^\omega\} \subset F$ (resp. $\{X_a : a \in [\lambda]^\omega_b\} \subset F$).

By $\lambda \to [\mu]^\omega_\kappa$ (resp. $\lambda \to [\mu]^{\omega, b}_\kappa$), we mean that for any $f : [\lambda]^\omega \to \kappa$ (resp. $f : [\lambda]^\omega_b \to \kappa$) there exists $x \in [\lambda]^\mu$ with $f''[x]^\omega \neq \kappa$ (resp. $f''[x]^\omega_b \neq \kappa$).

By $j : N \to M$, we mean that j is a nontrivial elementary embedding between transitive models $(N, \in)$ and $(M, \in)$ of ZFC. We also use i and k with subscripts other than j. V denotes the set-theoretic universe. By crit j (resp. fix j), we denote the least ordinal moved by j (resp. fixed by j above crit j).

Theorem 1.1 is essential in both of Kunen's and our proof of Theorem 1.

1.1. Theorem (Erdös-Hajnal [2]). *$\lambda \not\to [\lambda]^\omega_\lambda$ holds for any infinite cardinal λ.*

2. Proof of Theorem 1

We start by showing a simple fact on combinatorics of $[\lambda]^\lambda$, which is a subtheory of that of $P\lambda$ studied in [1, 9].

2.1. Proposition. *Any strongly normal filter on $[\lambda]^\lambda$ is principal.*

Proof: Let F be strongly normal. We show $\{\lambda\} \in F$.

Let $f : [\lambda]^\omega \to \lambda$ be a witness for $\lambda \not\to [\lambda]^\omega_\lambda$ i.e. $f''[x]^\omega = \lambda$ for any $x \in [\lambda]^\lambda$. Set $X_a = \{x \in [\lambda]^\lambda : x \ni f(a)\} \in F$ for $a \in [\lambda]^\omega$. Then $\Delta_{a \in [\lambda]^\omega} X_a = \{x \in [\lambda]^\lambda : f''[x]^\omega \subset x\} = \{\lambda\} \in F$. ∎

Before proceeding to the proof of Theorem 1, let us note the following simple fact.

2.2. Proposition (Foreman [3]). *Let $j : V \to M$ and $j''\lambda \in M$. Then $P\lambda \in M$.*

Now we prove Theorem 1.

Assume otherwise. Then λ is a cardinal by Proposition 2.2 and we can define an ultrafilter U on $[\lambda]^\lambda$ by "$X \in U$ iff $j''\lambda \in jX$". First we claim that U is strongly normal.

Fix $\{X_a : a \in [\lambda]^\omega\} \subset U$. Then $j''\lambda \in j(\Delta_{a\in[\lambda]^\omega} X_a) = \{x \in [\lambda]^\lambda \cap M : \forall a \in [x]^\omega \cap M \ x \in (jX)_a\}$, since $j''a = ja$ for any $a \in [\lambda]^\omega$. Thus $\Delta_{a\in[\lambda]^\omega} X_a \in U$.

Next we claim that U is non-principal.

Otherwise we can pick $x \in [\lambda]^\lambda$ with $jx = j''\lambda$. Then $x = \lambda$ and hence $\lambda = j''\lambda$. Contradiction.

Now that Proposition 2.1 and the two claims above yield the desired contradiction. ∎

Let us conclude this section with a remark, which leads us the embedding considered in the next section.

2.3. Theorem. *Let $j : V \to M$ and $\lambda = \mathit{fix}\, j$. Then $j''x \notin M$ for any $x \in (\lambda)^\lambda$.*

Before proceeding to the proof, let us observe the following simple fact.

2.4. Proposition. *Let $j : V \to M$. Then $j''x \subsetneq jx$ for any $x \subset Ord$ with ot $x \geq$ crit j.*

Proof: We show only properness of the inclusion.

Fix $\alpha \in x$ with $\mathrm{ot}\,(x \cap \alpha) = \kappa = \mathrm{crit}\, j$. Then $\mathrm{ot}\,(j''x \cap j\alpha) = \mathrm{ot}\, j''(x \cap \alpha) = \kappa$, while $\mathrm{ot}\,(jx \cap j\alpha) = \mathrm{ot}\, j(x \cap \alpha) = j\kappa$. Hence $j''x \neq jx$. ∎

Proof of Theorem 2.3: To argue by way of contradiction, fix $x \in (\lambda)^\lambda$ with $j''x \in M$. We show that for any $f : [x]^\omega \to x$ there exists $y \in (x)^\lambda$ with $f''[y]^\omega \subsetneq x$, which, together with Theorem 1.1 for $|\lambda|$, yields the desired contradiction.

Fix $f : [x]^\omega \to x$. Then $j''x$ witnesses $M \models$ "$\exists y \in (jx)^\lambda \ (jf)''[y]^\omega \subsetneq jx$", since $(jf)''[j''x]^\omega \subset j''x$ and by Proposition 2.4. Hence $\exists y \in (x)^\lambda \ f''[y]^\omega \subsetneq x$ by elementarity of j. ∎

3. Proof of Theorem 2

First of all, we give a characterization of the elementary embedding stated in Theorem 2 using an extender. In the course of the proof, we show that it is also equivalent to the axiom I2 considered in [5, 8].

The notion of an extender dates back to Powell [7]. Our proof is just a combination of his argument and one for huge embeddings in [8].

3.1. Theorem. *The following are equivalent.*

(1) *There exists $j : V \to M$ with $\kappa = \mathit{crit}\, j, \lambda = \mathit{fix}\, j$ and $V_\lambda \subset M$.*

(2) *There exists $j : V \to M$ with $\kappa = crit\ j, \lambda = fix\ j$ and $\{j''\alpha : \alpha < \lambda\} \subset M$.*
(3) *There exist an increasing sequence of cardinals $\{\kappa_n : n < \omega\}$ with $\kappa = \kappa_0$ and $\lambda = \sup_{n<\omega} \kappa_n$, and $\{U_n : n < \omega\}$ such that U_n is a strongly normal nonprincipal κ-complete ultrafilter on $[\kappa_{n+1}]^{\kappa_n}$ for any $n < \omega$, that $Y \in U_m$ iff $\{y \in [\kappa_{n+1}]^{\kappa_n} : y \cap \kappa_{m+1} \in Y\} \in U_n$ for any $m < n < \omega$ and $Y \subset [\kappa_{m+1}]^{\kappa_m}$, and that for any $\{Y_n : n < \omega\}$ with $Y_n \in U_n$ for any $n < \omega$ there exists $x \subset \lambda$ with $x \cap \kappa_{n+1} \in Y_n$ for any $n < \omega$.*

PROOF: (1)$\to$(2). Trivial.

(2)$\to$(3). Let $j : V \to M$ be as in (2). Define a cardinal κ_n inductively by $\kappa_0 = \kappa$ and $\kappa_{n+1} = j\kappa_n$, and an ultrafilter U_n on $[\kappa_{n+1}]^{\kappa_n}$ by "$Y \in U_n$ iff $j''\kappa_{n+1} \in jY$". Then it is straightforward to see $\lambda = \sup_{n<\omega} \kappa_n$ and strong normality and coherence of U_n's. Hence we show only the last clause in (3).

Assume otherwise. Then we can pick $\{Y_n : n < \omega\}$ such that $Y_n \in U_n$ for any $n < \omega$ and that for any $x \subset \lambda$ there exists $n < \omega$ with $x \cap \kappa_{n+1} \notin Y_n$. Set $T = \bigcup_{n<\omega}\{y \in [\kappa_{n+1}]^{\kappa_n} : \forall m \leq n\ y \cap \kappa_{m+1} \in Y_m\}$. Define a partial order $<_T$ on T by "$z <_T y$ iff $m < n$ and $y = z \cap \kappa_{m+1}$", where $y \in T \cap [\kappa_{m+1}]^{\kappa_m}$ and $z \in T \cap [\kappa_{n+1}]^{\kappa_n}$. We claim that $<_T$ is well-founded on T.

Otherwise we can pick $\{y_n : n < \omega\} \subset T$ with $y_{n+1} <_T y_n$ for any $n < \omega$. Set $y = \bigcup_{n<\omega} y_n$. Then $y \cap \kappa_{n+1} = y_n \cap \kappa_{n+1} \in Y_n$ for any $n < \omega$. Contradiction.

Let $r : T \to Ord$ be the rank function with respect to $<_T$, i.e. r is defined inductively by $r(y) = \sup\{r(z) + 1 : z <_T y\}$. Then for any $m < n < \omega$, $\bigcap_{l \leq n}\{y \in [\kappa_{n+1}]^{\kappa_n} : y \cap \kappa_{l+1} \in Y_l\} \subset \{y \in T \cap [\kappa_{n+1}]^{\kappa_n} : r(y) < r(y \cap \kappa_{m+1})\} \in U_n$ by the definition of $<_T$, and hence $(jr)(j''\kappa_{n+1}) < (jr)(j''\kappa_{n+1} \cap j\kappa_{m+1}) = (jr)(j''\kappa_{m+1})$ by the definition of U_n. Contradiction.

(3)$\to$(1). Let $\{\kappa_n : n < \omega\}$ and $\{U_n : n < \omega\}$ be as in (3). Let $i_n : V \to M_n = Ult(V, U_n)$ be the canonical embedding for $n < \omega$. Define an elementary embedding $i_{m,n} : M_m \to M_n$ by $i_{m,n}([f]_{U_m}) = [\langle f(x \cap \kappa_{m+1}) : x \in [\kappa_{n+1}]^{\kappa_n}\rangle]_{U_n}$ for $m \leq n < \omega$. Let M be the limit of the direct system $(\{M_n : n < \omega\}, \{i_{m,n} : m \leq n < \omega\})$, or equivalently, M is defined as follows.

Set $[\lambda]^\lambda_* = \{x \subset \lambda : \forall n < \omega\ |x \cap \kappa_{n+1}| = \kappa_n\}$. Define a subalgebra $P_f[\lambda]^\lambda_*$ of the boolean algebra $P[\lambda]^\lambda_*$ by "$X \in P_f[\lambda]^\lambda_*$ iff $X = \{x \in [\lambda]^\lambda_* : x \cap \kappa_{n+1} \in Y\}$ for some $n < \omega$ and $Y \subset [\kappa_{n+1}]^{\kappa_n}$". An ultrafilter U in $P_f[\lambda]^\lambda_*$ is defined by "$X \in U$ iff $X = \{x \in [\lambda]^\lambda_* : x \cap \kappa_{n+1} \in Y\}$ for some $n < \omega$ and $Y \in U_n$". Let $F_f[\lambda]^\lambda_*$ be the proper class of all

$g : [\lambda]^\lambda_* \to V$ such that there exist $n < \omega$ and $h : [\kappa_{n+1}]^{\kappa_n} \to V$ with $g(x) = h(x \cap \kappa_{n+1})$ for any $x \in [\lambda]^\lambda_*$. An equivalence relation $\sim_U$ on $F_f[\lambda]^\lambda_*$ is defined by "$g \sim_U g'$ iff $\{x \in [\lambda]^\lambda_* : g(x) = g'(x)\} \in U$". Set $M = \{[g]_U : g \in F_f[\lambda]^\lambda_*\}$, where $[g]_U$ is the equivalence class of g under $\sim_U$. A membership relation $\in_M$ on M is defined by "$[g]_U \in_M [g']_U$ iff $\{x \in [\lambda]^\lambda_* : g(x) \in g'(x)\} \in U$". First we claim that $\in_M$ is well-founded on M.

Otherwise there exists $\{g_n : n < \omega\} \subset F_f[\lambda]^\lambda_*$ with $\{x \in [\lambda]^\lambda_* : g_{n+1}(x) \in g_n(x)\} \in U$ for any $n < \omega$, or equivalently, there exist an increasing $\{n_m : m < \omega\} \subset \omega$ and $\{h_m : [\kappa_{n_m+1}]^{\kappa_{n_m}} \to V : n < \omega\}$ with $Y_{n_{m+1}} = \{y \in [\kappa_{n_{m+1}+1}]^{\kappa_{n_{m+1}}} : h_{m+1}(y) \in h_m(y \cap \kappa_{n_m+1})\} \in U_{n_{m+1}}$ for any $m < \omega$. Then we can pick $x \subset \lambda$ such that for any $m < \omega$, $x \cap \kappa_{n_{m+1}} \in Y_{n_{m+1}}$, i.e. $h_{m+1}(x \cap \kappa_{n_{m+1}+1}) \in h_m(x \cap \kappa_{n_m+1})$. Contradiction.

Hence we identify M with its transitive collapse and claim that $V_\lambda \subset M$.

Fix $n < \omega$. Then M_n is closed under κ_{n+1}-sequences, since $i_n'' \kappa_{n+1} = [id]_{U_n} \in M_n$. Hence κ_{n+1} is inaccessible, since $i_n(\kappa_m) = i_n([\langle |x| : x \in [\kappa_{m+1}]^{\kappa_m}\rangle]_{U_m}) = [\langle |x \cap \kappa_{m+1}| : x \in [\kappa_{n+1}]^{\kappa_n}\rangle]_{U_n} = |i_n''\kappa_{n+1} \cap i_n(\kappa_{m+1})| = |i_n''\kappa_{m+1}| = \kappa_{m+1}$ for any $m \leq n$ and since κ_0 is inaccessible. Thus $V_{\kappa_{n+1}} \subset M_n$.

Now let $k_n : M_n \to M$ be the canonical embedding. Then crit $k_n > \kappa_{n+1}$, since $i_{n,l}(\alpha) = i_{n,l}([\langle \text{ot } (x \cap \alpha) : x \in [\kappa_{n+1}]^{\kappa_n}\rangle]_{U_n}) = [\langle \text{ot } (x \cap \alpha) : x \in [\kappa_{l+1}]^{\kappa_l}\rangle]_{U_l} = \alpha$ for any $\alpha \leq \kappa_{n+1}$, i.e. crit $i_{n,l} > \kappa_{n+1}$ for any $n \leq l < \omega$. Thus $V_{\kappa_{n+1}} \subset M$. ∎

In Theorem 3.1(3), we adopt strong normality rather than normality just for later use.

3.2. Proposition. *Assume that $\lambda \not\to [\lambda]^{\omega,b}_\lambda$ holds. Then $\{\lambda\} \in F$ for any strongly seminormal filter F on $[\lambda]^\lambda$.*

The proof is analogous to that of Proposition 2.1.

3.3. Lemma. *Assume that the assertion (3) in Theorem 3.1 holds. Then there exist a strongly seminormal filter F on $[\lambda]^\lambda$ and $X \in F$ with $\lambda \notin X$.*

Proof: Define F by "$X \in F$ iff $\{x \in [\lambda]^\lambda : \forall n{<}\omega\ x \cap \kappa_{n+1} \in Y_n\} \subset X$ for some $\{Y_n : n < \omega\}$ with $Y_n \in U_n$ for any $n < \omega$", where $\{\kappa_n : n < \omega\}$ and $\{U_n : n < \omega\}$ are as in (3). First we claim that F is strongly seminormal.

Fix $Y_{a,n} \in U_n$ for $a \in [\lambda]^\omega_b$ and $n < \omega$. Set $X_a = \{x \in [\lambda]^\lambda : \forall n < \omega\ x \cap \kappa_{n+1} \in Y_{a,n}\} \in F$ for $a \in [\lambda]^\omega_b$. We show that $X = \Delta_{a \in [\lambda]^\omega_b} X_a \in F$.

Set $Z_{a.n} = \bigcap_{m \leq n} \{y \in [\kappa_{n+1}]^{\kappa_n} : y \cap \kappa_{m+1} \in Y_{a,m}\} \in U_n$ for $a \in [\lambda]^\omega_b$ and $n < \omega$. Then for any $n < \omega$, $Z_n = \Delta_{a \in [\kappa_{n+1}]^\omega} Z_{a,n} \in U_n$ by strong normality of U_n. Hence $Z = \{x \in [\lambda]^\lambda : \forall n < \omega \; x \cap \kappa_{n+1} \in Z_n\} \in F$. We show that $Z \subset X$ (in fact, equality holds) and complete the proof of strong seminormality of F.

Fix $x \in Z, a \in [\lambda]^\omega_b$ and $m < \omega$. Pick $m < n < \omega$ with $a \in [\kappa_{n+1}]^\omega$. Then $x \cap \kappa_{n+1} \in Z_{a,n}$, since $x \in Z$. Hence $x \cap \kappa_{m+1} \in Y_{a,m}$ by the definition of $Z_{a,n}$. Thus $x \in X$.

It is obvious that $\{x \in [\lambda]^\lambda : |x \cap \kappa_1| = \kappa_0\}$ witnesses the latter statement in the lemma. ∎

Now that Theorem 3.1, Proposition 3.2 and Lemma 3.3 prove Theorem 2.

We remark that the existence of a bounded ω-Jónsson function in Theorem 2 can be weakened as follows. Let F_λ be the minimal strongly seminormal filter on $[\lambda]^\lambda$. Then by the proof above, it suffices to assume that $\{\lambda\} \in F_\lambda$, which is equivalent to the existence of $f : [\lambda]^\omega_b \to \lambda$ such that the closure of x under f is λ for any $x \in [\lambda]^\lambda$ by Proposition 3.4.

3.4. PROPOSITION. *$X \in F_\lambda$ iff $\{x \in [\lambda]^\lambda : f''[x]^\omega_b \subset x\} \subset X \subset [\lambda]^\lambda$ for some $f : [\lambda]^\omega_b \to \lambda$.*

PROOF: We show only-if part, i.e. that the set $\{\{x \in [\lambda]^\lambda : f''[x]^\omega_b \subset x\} : f : [\lambda]^\omega_b \to \lambda\}$ generates a strongly seminormal filter.

Fix $f : [\lambda]^\omega_b \times [\lambda]^\omega_b \to \lambda$. Define $g : [\lambda]^\omega_b \to [\lambda]^{<\omega}$ by $g(a) = \{f(c,d), \pi(\alpha,\alpha), \pi(\alpha,\beta), \pi(\beta,\alpha)\}$ if $a = \pi''c \times d$ and $g(a) = \{\pi(\alpha,\alpha), \pi(\alpha,\beta), \pi(\beta,\alpha)\}$ otherwise, where $\pi : \lambda^2 \to \lambda$ is the canonical bijection, $\alpha = \min a$ and $\beta = \min(a - (\alpha+1))$. Then $\{x \in [\lambda]^\lambda : g''[x]^\omega_b \subset Px\} \subset \{x \in [\lambda]^\lambda : f''[x]^\omega_b \times [x]^\omega_b \subset x\}$, since $\pi''c \times d \in [x]^\omega_b$ for any $x \in [\lambda]^\lambda$ closed under g and $c, d \in [x]^\omega_b$.

Next define $h : [\lambda]^\omega_b \to \lambda$ by $h(a) = g_n(\{\gamma_\xi : \xi < \alpha\})$, where $\pi : (\omega_1 - \omega) \times \omega \to \omega_1 - \omega$ is a bijection with $\pi(\alpha, n) \geq \alpha$ for any α and n, $\{\gamma_\xi : \xi < \pi(\alpha,n)\}$ is the increasing enumeration of a and $\{g_m(a) : m < \omega\} = g(a)$. Then $\{x \in [\lambda]^\lambda : h''[x]^\omega_b \subset x\} \subset \{x \in [\lambda]^\lambda : g''[x]^\omega_b \subset Px\}$, since for any $a \in [x]^\omega_b$ and $n < \omega$ there exists $c \in [x]^\omega_b$ with $h(c) = g_n(a)$. ∎

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