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SOME FUNCTION-THEORETIC PROPERTIES OF THE GAUSS MAP OF MINIMAL SURFACES

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§1. Introduction.

According to the classical Liouvill's theorem, there is no bounded nonconstant holomorphic function on the complex plane $\mathbf{C}$. On the other hand, the classical Bernstein's theorem asserts that there is no nonflat minimal surface in $\mathbf{R}^3$ which is described as the graph of a C^2-function on $\mathbf{R}^2$. The conclusions of these two theorems have a strong resemblance and they are closely related. Liouville's theorem was improved as Casoratti-Weierstrass theorem, Picard's theorem and Nevanlinna theory, which were generalized to the case of holomorphic curves in the projective space $P^n(\mathbf{C})$ by E. Borel, H. Cartan, J. and H. Weyl and L. V. Ahlfors. On the other hand, Bernstein's theorem was improved by many researchers in the field of differential geometry, Heinz, Hopf, Nitsche, Osserman, Chern and so on. As for recent results, H. Fujimoto proved that the Gauss map of complete mimimal surfaces in $\mathbf{R}^3$ can omit at most four values([5]). He obtained also modified defect relations for the Gauss map of complete minimal surfaces in $\mathbf{R}^m$ which have analogies to the defect relation in Nevanlinna theory([6], [7], [8]). Related to these subjects there are several results which were obtained by X. Mo and R. Osserman, S. J. Kao, M. Ru and so on([14], [13], [20]). Moreover, H. Fujimoto gave the curvature estimates of minimal surfaces related to execeptional values of the Gauss maps ([9]), and obtained some unicity theorems which are analogies to Nevanlinna's unicity theorem for meromorphic functions ([10]). In this lecture, we expose some of these function-theoretic properties of the Gauss map of minimal surfaces in $\mathbf{R}^m$.

§2. Minimal surfaces and their Gauss maps.

Consider an (oriented) surface $x = (x_1, \dots, x_m) : M \to \mathbf{R}^m$ immersed in $\mathbf{R}^m$. Taking a holomorphic local coordinate $z := u + \sqrt{-1}v$ associated with each positively oriented isothermal coordinates (u, v), M is regarded as a Riemann surface with a conformal metric. To explain the Gauss map of M, consider the set Π of all oriented 2-planes in $\mathbf{R}^m$ which contain the origin. For each $P \in \Pi$ taking a positively oriented basis $\{X, Y\}$ of P such that $|X| = |Y|$, $(X,\ Y) = 0$ and setting $\Phi(P) := \pi(X - \sqrt{-1}Y) \in P^{m-1}(\mathbf{C})$ we define the map $\Phi : \Pi \to P^{m-1}(\mathbf{C})$, where $\pi : \mathbf{C}^m - \{0\} \to P^{m-1}(\mathbf{C})$ denotes the canonical projection. This is well-defined. Because, for another positively oriented basis $\{\tilde{X}, \tilde{Y}\}$ of P with $|\tilde{X}| = |\tilde{Y}|$, $(\tilde{X},\ \tilde{Y}) = 0$ there are some real numbers r and θ such that $\tilde{X} - \sqrt{-1}\tilde{Y} = re^{i\theta}(X - \sqrt{-1}Y)$, so that $\pi(X - \sqrt{-1}Y) = \pi(\tilde{X} - \sqrt{-1}\tilde{Y})$. On the other hand, $\Phi(P)$ is contained in the quadric

$$Q_{m-2}(\mathbf{C}) := \{(w_1 : \dots : w_m); w_1^2 + \dots + w_m^2 = 0\}(\subset P^{m-1}(\mathbf{C})).$$

In fact, for the above basis $\{X, Y\}$ of P we have

$$(X - \sqrt{-1}Y,\ X - \sqrt{-1}Y) = (X, X) - 2\sqrt{-1}(X, Y) - (Y, Y) = 0.$$

Moreover, we can easily show that the map $\Phi : \Pi \to Q_{m-2}(\mathbf{C})$ is bijective.

For a surface $x = (x_1, x_2, \cdots, x_m) : M \to \mathbf{R}^m$ immersed in $\mathbf{R}^m$, we define the Gauss map of M as the map G of M into $Q_{m-2}(\mathbf{C})$ which maps each point $a \in M$ to the Φ-image of the oriented tangent plane of

M at a. Usually, the conjugate of the map G is called the Gauss map of M. We adopt the above definition for convenience' sake for simplifying the description of function-theoretic properties of minimal surfaces.

For a system of positive isothermal local coordinates (u, v), since the vectors $X = \partial x/\partial u$ and $Y = \partial x/\partial v$ satisfy the conditions $|X| = |Y|$ and $(X, Y) = 0$, the Gauss map of M is locally given by

$$G = \pi(X - \sqrt{-1}Y) = \left(\frac{\partial x_1}{\partial z} : \frac{\partial x_2}{\partial z} : \cdots : \frac{\partial x_m}{\partial z}\right),$$

where $z = u + \sqrt{-1}v$. We may write $G = (\omega_1 : \cdots : \omega_m)$ with globally defined forms $\omega_i := \partial x_i \equiv (\partial x_i/\partial z)dz$.

By definition, M is a minimal surface if the mean curvature of M for every normal direction vanishes everywhere. This is equivalent to the condition that each component x_i of x is harmonic on M with respect to isothermal coordinates. We have the following criterion for minimal surfaces.

Proposition 2.1. *A surface $x : M \to \mathbf{R}^m$ is minimal if and only if the Gauss map $G : M \to P^{m-1}(\mathbf{C})$ is holomorphic.*

In fact, if M is minimal, then each ω_i, and so G, is holomorphic because $\bar{\partial}\omega_i = \bar{\partial}\partial x_i d\bar{z} \wedge dz = 0$. For the proof of the converse, refer to [12, Theorem 1.1].

We say that a holomorphic form ω on a Riemann surface M has no real period if $\mathrm{Re} \int_\gamma \omega = 0$ for every closed piecewise smooth continuous curve γ in M. If ω has no real period, then $x(z) = \mathrm{Re} \int_{\gamma_{z_0}^z} \omega$ depends only on z and z_0 for a piecewise smooth continuous curve $\gamma_{z_0}^z$ in M joining z_0 and z and hence x is a single-valued function on M. We can easily show the following construction theorem of minimal surfaces.

Theorem 2.2. *Let M be an open Riemann surface and let ω_i $(1 \leq i \leq m)$ be holomorphic forms on M with local expressions $\omega_i = f_i dz$ such that they have no common zero, no real periods and satisfy the identity $\sum_{i=1}^m f_i^2 = 0$. Then, for an arbitrarily fixed $z_0 \in M$ the surface $x = (x_1, \ldots, x_m) : M \to \mathbf{R}^m$ defined by the functions $x_i = 2 \operatorname{Re} \int_{z_0}^{z} \omega_i$ is a minimal surface immersed in $\mathbf{R}^m$ whose Gauss map is the map $G = (\omega_1 : \cdots : \omega_m) : M \to Q_{m-2}(\mathbf{C})$.*

For the particular case $m = 3$, to each $P \in \Pi$ there corresponds the unique positively oriented normal unit vector $N(\in S^2)$ of P, which determines a point in $\bar{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ through the stereographic projection ϖ. In this lecture, we define the classical Gauss map of M as the map G which maps each $a \in M$ to the positively oriented unit normal vector of M at a. In some cases, we call the map $g := \varpi \cdot G$ the classical Gauss map of M. It is shown that the classical Gauss map g is locally represented as

$$g = \frac{f_3}{f_1 - \sqrt{-1} f_2}$$

with the functions $f_i := \partial x_i / \partial z$ $(1 \leq i \leq 3)$ ([19]). As a result of Proposition 2.1, a surface M immersed in $\mathbf{R}^3$ is minimal if and only if the classical Gauss map g is a meromorphic function on M.

We explain here the Enneper-Weierstrass representation theorem of minimal surfaces, which is a restatement of Theorem 2.2 for the particular case $m = 3$.

Theorem 2.3. *Let $x = (x_1, x_2, x_3) : M \to \mathbf{R}^3$ be a nonflat minimal surface immersed in $\mathbf{R}^3$ and g the classical Gauss map of M. Set $\omega_i := \partial x_i$ $(1 \leq i \leq 3)$ and $\omega := \omega_1 - \sqrt{-1}\omega_2$. Then,*

$$\omega_1 := \frac{1 - g^2}{2}\omega, \ \omega_2 := \frac{\sqrt{-1}(1 + g^2)}{2}\omega, \ \omega_3 := g\omega, \tag{2.4}$$

and the metric is given by $ds^2 = (1 + |g|^2)^2|\omega|^2$. *Moreover,* ω *has a zero of order* $2k$ *when and only when* g *has a pole of order* k.

Conversely, if we take an open Riemann surface M, *nonzero holomorphic form* ω *and a nonconstant meromorphic function* g *on* M *such that* ω *has a zero of order* $2k$ *when and only when* g *has a pole of order* k *and the holomorphic forms* ω_i $(1 \leq i \leq 3)$ *defined by* (2.4) *have no real periods, then the functions*

$$x_i := 2\mathrm{Re} \int^z \omega_i \quad (1 \leq i \leq 3) \tag{2.5}$$

define a minimal surface $x = (x_1, x_2, x_3) : M \to \mathbf{R}^3$ *immersed in* $\mathbf{R}^3$ *whose classical Gauss map is the map* g.

For the proofs of Theorems 2.3, refer to [19, p. 64].

§3. The Gaussian curvature of minimal surfaces.

In 1952, E. Heinz showed that, for a minimal surface M in $\mathbf{R}^3$ which is the graph of a function $z = z(x,y)$ of class C^2 defined on a disk $\Delta_R := \{(x,y); x^2+y^2 < R^2\}$, there is a positive constant C not depending on each surface M such that $|K(0)| \leq C/R^2$, where we denote by $K(a)$ the curvature of M at a ([11]). This is an improvement of the classical Bernstein's theorem. For, if $z(x,y)$ is a function on $\mathbf{R}^2$, we may take an arbitrary point in $\mathbf{R}^2$ as the origin after a coordinate change and $R = \infty$, so that M is necessarily flat. Later, R. Osserman obtained some related results. One of them is stated as follows([17]):

Theorem 3.1. *Let* M *be a simply-connected minimal surface immersed in* $\mathbf{R}^3$ *and assume that there is some fixed nonzero vector* n_0 *and a number* $\theta_0 > 0$ *such that all normals to* M *make angles of at least* θ_0

with n_0. Then, it holds that

$$|K(a)|^{1/2} \leq \frac{1}{d(a)} \frac{2\cos(\theta_0/2)}{\sin^3(\theta_0/2)} \qquad (a \in M),$$

where $d(a)$ denotes the distance from a to the boundary of M.

In connection with these results, H. Fujimoto proved the following theorem in his paper [5].

Theorem 3.2. *Let M be a minimal surface immersed in $\mathbf{R}^3$ and let $G : M \to S^2$ be the classical Gauss map of M. If G omits mutually distinct five points $n_1, \ldots, n_5$ in S^2, then it holds that*

$$|K(a)|^{1/2} \leq \frac{C}{d(a)} \qquad (a \in M)$$

for some positive constant C depending only on n_j's.

In Theorem 3.2, if M is complete, then $d(a) = \infty (a \in M)$ and so M is necessarily flat. Therefore, the classical Gauss map of a complete non-flat minimal surface immersed in $\mathbf{R}^3$ can omit at most four points. Here, the number four is best-possible. In fact, there are many examples of nonflat complete minimal surfaces immersed in $\mathbf{R}^3$ whose classical Gauss maps omit exactly four values([19]). Among them, Scherk's surface is most famous.

Recently, he gave the following more precise estimate of the Gaussian curvature of minimal surfaces([9]).

Theorem 3.3. *Let $x = (x_1, x_2, x_3) : M \to \mathbf{R}^3$ be a minimal surface immersed in $\mathbf{R}^3$ and let $G : M \to S^2$ be the Gauss map of M. Assume that G omits five distinct points $n_1, \ldots, n_5 \in S^2$. Let θ_{ij} be the angle between n_i and n_j and set*

$$L := \min\left\{\sin\left(\frac{\theta_{ij}}{2}\right); 1 \leq i < j \leq 5\right\}.$$

Then, there exists some positive constant C not depending on each minimal surface such that

$$(3.4) \qquad |K(a)|^{1/2} \leq \frac{C}{d(a)} \frac{\log^2 \frac{1}{L}}{L^3} \qquad (a \in M).$$

These theorems are proved by the use of the generalized Schwarz' lemma obtained by L. Ahlfors([1]) and the negative curvature method which is used by Cowen-Griffiths in the proof of the second main theorem for holomorphic curves in $P^n(\mathbf{C})$(cf., [4]).

Related to Theorem 3.3, for an arbitrarily given $\varepsilon > 0$ we can give an example of a family of minimal surfaces which shows that there is no positive constant C not depending on each minimal surface which satisfies the condition

$$(3.5) \qquad |K(a)|^{1/2} \leq \frac{C}{d(a)} \frac{1}{L^{3-\varepsilon}}.$$

To show this, for each positive number $R (\geq 1)$ we take five points

$$\alpha_1 := R, \ \alpha_2 := \sqrt{-1}R, \ \alpha_3 := -R, \ \alpha_4 := -\sqrt{-1}R, \ \alpha_5 := \infty$$

in $\bar{\mathbf{C}}$. Taking the form $\omega = dz$ and the function $g(z) = z$, we define the surface $x = (x_1, x_2, x_3) : \Delta_R := \{z; |z| < R\} \to \mathbf{R}^3$ with the use of the functions x_i $(1 \leq i \leq 3)$ given by (2.4) and (2.5). Then, by Theorem 2.3 this is a minimal surface immersed in $\mathbf{R}^3$ whose Gauss map is g and whose metric is given by $ds^2 = (1 + |z|^2)^2 |dz|^2$. We easily have

$$d(0) = \int_0^R (1 + x^2) dx = R + \frac{1}{3} R^3$$

and $|K(0)|^{1/2} = 2$. On the other hand, the quantity L for the points in S^2 corresponding to α_j's is given by $L = 1/\sqrt{1 + R^2}$ and so

$$|K(0)|^{1/2} d(0) L^{3-\varepsilon} = \frac{2(R + \frac{1}{3}R^3)}{(1 + R^2)^{(3-\varepsilon)/2}},$$

which converges to $+\infty$ as R tends to $+\infty$. Therefore, there is no positive constant satisfying the condition (3.5) which does not depend on each minimal surface.

It is a very interesting open problem to know whether the factor $\log^2\left(\frac{1}{L}\right)$ in (3.4) can be removed or not.

§4. Unicity theorems for the Gauss maps of minimal surfaces.

In 1926, R. Nevanlinna gave the following unicity theorem of meromorphic functions as an application of the second main theorem in his value distribution theory for meromorphic functions([15]).

Theorem 4.1. *Let φ and ψ be nonconstant meromorphic functions on* $\mathbf{C}$. *If there are distinct five values $\alpha_1, \ldots, \alpha_5$ such that $\varphi^{-1}(\alpha_j) = \psi^{-1}(\alpha_j)$ for $j = 1, \ldots, 5$, then $\varphi \equiv \psi$.*

We can prove some unicity theorems for the Gauss map of complete minimal surfaces which are similar to Theorem 4.1. To state them, we consider two nonflat minimal surfaces

$$x := (x_1, \ldots, x_m) : M \to \mathbf{R}^m, \quad \tilde{x} := (\tilde{x}_1, \ldots, \tilde{x}_m) : \tilde{M} \to \mathbf{R}^m$$

such that there is a conformal diffeomorphism Φ between M and $\tilde{M}$. Let G and $\tilde{G}$ be the Gauss maps of M and $\tilde{M}$ respectively. Then the Gauss map of the minimal surface $\tilde{x} \cdot \Phi : M \to \mathbf{R}^m$ is given by $\tilde{G} \cdot \Phi$.

Set $N := m - 1$ and

$$f := G, \quad g := \tilde{G} \cdot \Phi,$$

and assume that they satisfy the following:

Assumption 4.2 . *There exist hyperplanes $H_1, \dots, H_q$ in $P^N(\mathbf{C})$ located in general position such that*

(i) $f(M) \not\subseteq H_j$ *and* $g(M) \not\subseteq H_j$ *for every* j.

(ii) $f = g$ *on the set* $\cup_{j=1}^{q} f^{-1}(H_j) \cup g^{-1}(H_j)$ *outside a compact subset* K *of* M.

Theorem 4.3. *Under Assumption 4.2, we have necessarily* $f \equiv g$

(A) *if* $q > m^2 + m(m-1)/2$ *for the case where* M *is complete and has infinite total curvature or*

(B) *if* $q \geq m^2 + m(m-1)/2$ *for the case where* $K = \emptyset$ *and both of* M *and* $\tilde{M}$ *are complete and have finite total curvature.*

For the classical Gauss maps of complete minimal surfaces in $\mathbf{R}^3$, we can prove the following:

Theorem 4.4. *Let* $x : M \to \mathbf{R}^3$ *and* $\tilde{x} : \tilde{M} \to \mathbf{R}^3$ *be complete minimal surfaces such that there is a conformal diffeomorphism* Φ *between* M *and* $\tilde{M}$. *Denote by* f *the classical Gauss map of* M *and by* g *the composite of* Φ *and the classical Gauss map of* $\tilde{M}$. *Assume that there exist distinct values* $\alpha_1, \dots, \alpha_q$ *such that* $f^{-1}(\alpha_j) = g^{-1}(\alpha_j)$ $(1 \leq j \leq q)$. *Then we have necessarily* $f \equiv g$

(C) *if* $q \geq 7$ *for the case where* M *is complete and has infinite total curvature or*

(D) *if* $q \geq 6$ *for the case where both of* M *and* $\tilde{M}$ *are complete and have finite total curvature.*

Here is an example which shows that the number seven in the above result for the case (D) is best-possible. To state this, take a number α with $\alpha \neq \pm 1, 0$ and consider

$$\omega := \frac{dz}{z(z-\alpha)(z-1/\alpha)}, \quad g_1(z) := z$$

and the universal covering surface M of $\mathbf{C} - \{0, \alpha, 1/\alpha\}$. Using these ω and g_1 we define the functions $x_i (1 \leq i \leq 3)$ by the formulas (2.4) and (2.5) and construct a minimal surface $x = (x_1, x_2, x_3) : M \to \mathbf{R}^3$ in $\mathbf{R}^3$. It is easily seen that M is complete. On the other hand, if we construct another minimal surface $y := (y_1, y_2, y_3) : \tilde{M} \to \mathbf{R}^3$ in the similar manner by the use of

$$\omega := \frac{dz}{z(z-\alpha)(z-1/\alpha)}, \quad g_2(z) = \frac{1}{z},$$

we can easily check that $\tilde{M}$ is isometric with M, so that the identity map $\Phi : z \in M \mapsto z \in \tilde{M}$ is a conformal diffeomorphism. For the maps g_1 and g_2 we have $g_1 \not\equiv g_2$ and $g_1^{-1}(\alpha_j) = g_2^{-1}(\alpha_j)$ for six values

$$\alpha_1 := 0, \ \alpha_2 := \infty, \ \alpha_3 := 1, \ \alpha_4 := -1 \ \alpha_5 := \alpha, \ \alpha_6 := \frac{1}{\alpha}.$$

These show that the number seven in Theorem 4.4 cannot be replaced by six.

§5. Modified defect relations for the Gauss map of minimal surfaces.

In 1929, R. Nevanlinna gave the defect relation as a reformulation of his second main theorem and it was generalized to the case of holomorphic curves in $P^n(\mathbf{C})$ by H. Cartan, J. and H. Weyl and L. Ahlfors. As an analogy, we can prove the modified defect relation for the Gauss map of complete minimal surfaces in $\mathbf{R}^m$, which will be explained in the followings.

Let M be an open Riemann surface with a conformal metric ds^2 and consider a nondegenerate holomorphic map f of M into $P^n(\mathbf{C})$. We mean by a divisor on M a map $\nu : M \to \mathbf{R}$ whose support has no accumulation point in M. Each divisor ν is considered as a (1.1)-current, which we denote by $[\nu]$. For a hyperplane H in $P^n(\mathbf{C})$ we consider the divisor $\nu(f, H)$ whose values at $a \in M$ is defined as the intersection multiplicity of H and the image of f at $f(a)$, and we set $f^*(H)^{[n]} := \min(\nu(f, H), n)$. We denote by Ω_f the pull-back of Fubini-Study metric on $P^n(\mathbf{C})$ through f. Occationally, these are regarded as (1, 1)-currents on M.

We define the modified defect of H for f by

$$D_f(H) := 1 - \inf\{\eta > 0; f^*(H)^{[n]} \prec \eta\Omega_f \text{ on } M - K \\ \text{for some compact set } K\}.$$

Here, by $\Omega_1 \prec \Omega_2$ we mean that there are a divisor ν and a bounded real-valued function k with mild singularities, in the meaning stated in [8, Definition 4.1], such that $\nu \geq c$ on the support of ν for a positive constant c and

$$\Omega_1 + [\nu] = \Omega_2 + dd^c \log |k|^2$$

holds as currents.

We also define the order of f by

$$\rho_f := \inf\{\rho > 0; -\operatorname{Ric}_{ds^2} \prec \rho\Omega_f \text{ on } M - K \text{ for some compact set } K\}.$$

After Chen[2] we say that hyperplanes $H_j (1 \leq j \leq q)$ are located in N-subgeneral position if $H_{j_0} \cap \ldots \cap H_{j_N} = \emptyset$ for all $1 \leq j_0 < \cdots < j_N \leq q$, where $q > N \geq n$. Particularly, we say that $H_j (1 \leq j \leq q)$ are in general position if they are in n-subgeneral position.

The modified defect relation for holomorphic curves in $P^n(\mathbf{C})$ is stated as follows :

Theorem 5.1. *Let M be an open Riemann surface with a complete conformal metric ds^2 and let f be a nondegenerate holomorphic map of*

M into $P^n(\mathbf{C})$. For the particular case where M is biholomorphic with a compact Riemann surface $\bar{M}$ with finitely many points removed, assume that f cannot be extended to a holomorphic map of $\bar{M}$ into $P^n(\mathbf{C})$. Then, for arbitrary hyperplanes $H_1, \ldots, H_q$ in $P^n(\mathbf{C})$ located in N-subgeneral position,

$$\sum_{j=1}^{q} D_f(H_j) \leq (2N - n + 1)\left(1 + \frac{\rho_f n}{2}\right).$$

Let M be a minimal surface immersed in $\mathbf{R}^m$. The surface M is reegarded as an open Riemann surface with conformal metric and we can easily show that $\rho_G \leq 1$ for the Gauss map G. On the other hand, it is known that a complete minimal surface M immersed in $\mathbf{R}^m$ has finite total curvature if and only if M is biholomorphic with a compact Riemann surface $\bar{M}$ with finitely many points removed and the Gauss map is holomorphically extended to $\bar{M}$ ([3]). Using these fact, we can conclude from Theorem 5.1 the following :

Theorem 5.2. *Let M be a nonflat complete minimal surface immersed in $\mathbf{R}^m$ with infinite total curvature and G the Gauss map of M. Then, for arbitrary hyperplanes $H_1, \ldots, H_q$ in $P^{m-1}(\mathbf{C})$ located in general position,*

$$\sum_{j=1}^{q} D_G(H_j) \leq \frac{m(m+1)}{2}.$$

For a holomorphic map f of an open Riemann surface M into $P^n(\mathbf{C})$ and a hyperplane H in $P^n(\mathbf{C})$ we can show that $D_f(H) = 1$ if $f^{-1}(H)$ is finite. This yields the following improvement of a result of Ru([20]):

Corollary 5.3. *Let M be a nonflat complete minimal surface immersed in $\mathbf{R}^m$ with infinite total curvature, and let G be the Gauss map of M. If $G^{-1}(H_j)$ are finite for q hyperplanes $H_1, \ldots, H_q$ in $P^{m-1}(\mathbf{C})$ located in general position, then $q \leq m(m+1)/2$.*

We can also apply Theorem 5.1 to the classical Gauss map g of complete minimal surface in $\mathbf{R}^3$. In this case, it is shown that $\rho_g \leq 2$. We have the following modified defect relation:

Theorem 5.4. *Let M be a nonflat complete minimal surface with infinite total curvature and let $g : M \to P^1(\mathbf{C})$ be the classical Gauss map. Then, for arbitrary distinct points $\alpha_1, \alpha_2, \ldots, \alpha_q$ in $P^1(\mathbf{C})$,*

$$\sum_{j=1}^{q} D_g(\alpha_j) \leq 4.$$

Since $D_g(\alpha) = 1$ when $\#g^{-1}(\alpha) < \infty$, we can conclued the folowing result of X. Mo and R. Osserman([14]).

Corollary 5.5. *Let M be a nonflat complete minimal surface with infinite total curvature immersed in $\mathbf{R}^m$. Then there are at most four distinct points in $P^1(\mathbf{C})$ whose inverse images by the classical Gauss map are finite.*

It is known that the Gauss map of a nonflat complete minimal surface in $\mathbf{R}^3$ with finite total curvature can omit at most three distinct values in $P^1(\mathbf{C})$([18]). The main result of [9] stated in §1 is an immediate consequence of Theorem 5.4.

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