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ON MORDELL-WEIL GROUPS OF ABELIAN SCHEMES

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§1 Mordell-Weil group.

Let S be a connected smooth quasi-projective variety defined over the field of complex numbers $\mathbf{C}$. An abelian scheme

$$f : A \longrightarrow S$$

is a smooth projective group scheme over S with connected fiber. For every closed point $s \in S$, the fiber $A_s = f^{-1}(s)$ is an abelian variety defined over $\mathbf{C}$.

(1.1) Definition. *For an abelian scheme $f : A \longrightarrow S$, we define the Mordell-Weil group $MW(A/S)$ by the group $A_\eta(K)$ of K-rational points of the generic fiber A_η, where $K = \mathbf{C}(S)$ is the function field of S and η denote the generic point of S.*

By definition, the Mordell-Weil group $MW(A/S)$ is isomorphic to the group

$$\{s : S \cdots \rightarrow A \text{ rational section of } f\},$$

and Hartogs' theorem and GAGA imply that this group is isomorphic to the group of regular sections of f.

(1.2) Definition. *Let K be the function field of S, and A_K an abelian variety defined over K. A $K/\mathbf{C}$-trace of A_K is a pair (B, τ) consisting of an abelian variety B defined over $\mathbf{C}$ and a homomorphism*

$$\tau : B \longrightarrow A_K$$

defined over K which has the following universal property. Given an abelian variety C defined over $\mathbf{C}$ and a homomorphism $\phi : C \longrightarrow A_K$, then there exists a unique homomorphism $\phi_* : C \longrightarrow B$ defined over $\mathbf{C}$ such that $\phi = \tau\phi_*$.

The existence of $(K/\mathbf{C})$-trace is proved by Chow. Moreover we have the following fundamental result due to Lang and Néron.

(1.3) Theorem-Definition. *(cf. Lang-Néron [La, p139, Th.2]) Let K be the function field of S, A_K an abelian variety defined over K, and (B,τ) a $(K/\mathbf{C})$-trace of A_K. Then $A(K)/\tau(B(\mathbf{C}))$ is finitely generated. If we write as*

$$A_K/\tau(B(\mathbf{C})) \simeq \mathbf{Z}^r \oplus (\mathit{Torsion}),$$

we call the rank r of the free part of $A_K/\tau(B(\mathbf{C}))$ the Mordell-Weil rank.

This is a function field analogue of Mordell-Weil theorem for abelian variety defined over a number field. In the number field case, there is a beautiful conjecture due to Birch & Swinnerton-Dyer about the relation between the order of zero of L-function and the Mordell-Weil rank. On the other hand, Mordell-Weil groups admit height pairings, which have been deeply studied by Shioda [Sd 2] and Cox-Zucker [C-Z] for elliptic surfaces and recently for families of Jacobian of curves (see [Sd3]). In this note, we will give a Hodge theoretic interpretation of Mordell-Weil groups by using Zucker's relative Hodge theory. We will also give a few applications in [Sa.MH] and recent result on the group of the component of the Néron model.

§2 Relative Hodge theory after Deligne and Zucker.

We shall give a Hodge theoretic interpretation of a Mordell-Weil group $MW(A/S)$. Given an abelian scheme $f : A \longrightarrow S$, let $\mathbf{R}_1 f_* \mathbf{Z}_X$ denote the local system of the first homology of fibers of f. Then from the relative exponential sequence we have the exact sequence of sheaves on S^{an}

$$0 \longrightarrow \mathbf{R}_1 f_* \mathbf{Z} \longrightarrow \mathcal{L}ie_{A/S} \longrightarrow \mathcal{O}_S^{an}(A) \longrightarrow 0. \tag{2.1}$$

Setting $\mathbf{V_Z} = \mathbf{R}_1 f_* \mathbf{Z}_X$ and using the isomorphism

$$\mathcal{L}ie_{A/S} \simeq R^1 f_* \mathcal{O}_A^{an},$$

we have the following exact sequence:

$$\begin{array}{ll} 0 \longrightarrow H^0(S, \mathbf{V_Z}) \xrightarrow{p_0} H^0(S, R^1 f_* \mathcal{O}_A^{an}) \longrightarrow H^0(S, \mathcal{O}_S^{an}(A)) \\ \xrightarrow{\delta} H^1(S, \mathbf{V_Z}) \xrightarrow{p_1} H^1(S, R^1 f_* \mathcal{O}_A^{an}) \qquad , \end{array} \tag{2.2}$$

It is well-known that the exact sequence (2.1) is equivalent to giving data of variation of Hodge structure (VHS) of weight (-1), moreover it is polarized by a relative ample line bundle on A.

(2.3) Definition. A polarized variation of Hodge structure (VHS) of weight -1 and of types (-1, 0), (0, -1) over S is data $(\mathbf{V_Z}, \mathbf{A}, \mathcal{F}^0)$ consisting of:

(i) a local system of free $\mathbf{Z}$-modules on S,
(ii) a flat $\mathbf{Z}$-valued non-degenerate symplectic form $\mathbf{A}$ on $\mathbf{W_Z}$,
(iii) and a locally free subsheaf $\mathcal{F}^0 \subset \mathcal{F}^{-1} := \mathbf{V_Z} \otimes_{\mathbf{Z}} \mathcal{O}_S$ such that

$$\mathcal{F}^0 \oplus \overline{\mathcal{F}^0} \cong \mathbf{V_Z} \otimes_{\mathbf{Z}} \mathcal{O}_S,$$

satisfying that

(HRBR): for any non-zero local section $u \in \mathcal{F}^0$, we have

$$\mathbf{A}(u, u) = 0,$$
$$-(\sqrt{-1})\mathbf{A}(u, \overline{u}) > 0,$$

(GT): and the Griffiths' transversality

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega_S.$$

Note that the subsheaf $\mathcal{F}^0 \subset \mathbf{V_Z} \otimes \mathcal{O}_S$ is given by

$$\ker(\mathbf{V_Z} \otimes \mathcal{O}_S \longrightarrow \mathcal{L}ie_{A/S} \simeq R^1 f_* \mathcal{O}_A^{an}),$$

hence one has

$$R^1 f_* \mathcal{O}_A^{an} \simeq Gr_{\mathcal{F}}^{-1}.$$

For a general polarized VHS, we have the following

(2.4) Theorem. *Let $\mathbf{V_Z}$ be a polarized VHS over S of weight m. Assume that S^{an} is compact. Then the cohomology group $H^q(S, \mathbf{V_Z})$ admits a Hodge structure of weight $q + m$ and a primitive decomposition.*

(2.5) Remark. This theorem is a starting point of studies of Hodge structures on the cohomology groups with coefficient in VHS. When dim $S = 1$ but S may be non-compact, then Zucker extended Deligne's result to the cohomology groups $H^q(\overline{S}, j_* \mathbf{V_Z})$, which is isomorphic to an intersection cohomology group $IH^q(S, \mathbf{V_Z})$. Now these kind of results have been extended to more general cases. (see [K-K], [Sa.Mo1, 2]).

Let $f : A \longrightarrow S$ be an abelian scheme and $(\mathbf{V_Z}, \mathbf{A}, \mathcal{F}^0)$ the corresponding polarized VHS. We define the filtration on the holomorphic de Rham complex $\Omega^{\cdot}_S(\mathbf{V_C})$ by

$$F^r \Omega_S^p(\mathbf{V_C}) = \Omega_S^p \otimes \mathcal{F}^{r-p}.$$

(Griffiths' transversality assures that they actually form subcomplexes.) Assume that S is compact. Then we have an isomorphism

$$H^n(S, \mathbf{V_C}) \cong H^n_{(2)}(S, \mathbf{V_C}) \quad \text{for all } n. \tag{2.6}$$

By using L^2-harmonic theory, one can show that there exists a Hodge decomposition

$$H^n(S,\mathbf{V_C}) \cong H^n_{(2)}(S,\mathbf{V_C}) = \oplus_{p+q=n-1} H^{p,q}. \tag{2.7}$$

Since one has a quasi-isomorphism $\mathbf{V_C} \simeq \Omega^{\cdot}_S(\mathbf{V_C})$,
the filtration $F^r\Omega^{\cdot}_S(\mathbf{V_C})$ induces a filtration on $H^n(S,\mathbf{V_C})$ and the Hodge components are given by

$$H^{p,q} \simeq \mathbf{H}^n(S, Gr^p_F\Omega^{\cdot}_S(\mathbf{V_C})).$$

For example, $H^0(S,\mathbf{V_C})$ has a 2-step filtration $0 = F^1 \subset F^0 \subset F^{-1}$ whose successive quotients are:

$$H^{0,-1} = Gr^0_F = F^0 = \mathbf{H}^0(\mathcal{F}^0 \longrightarrow \Omega_S \otimes Gr^{-1}_{\mathcal{F}}),$$
$$H^{-1,0} = Gr^{-1}_F = F^{-1}/F^0 = \mathbf{H}^0(Gr^{-1}_{\mathcal{F}}).$$

where $Gr^{-1}_{\mathcal{F}} = \mathcal{F}^{-1}/\mathcal{F}^0$. $H^1(S,\mathbf{V_C})$ has a 3-step filtration $0 = F^2 \subset F^1 \subset F^0 \subset F^{-1} = H^1$ whose successive quotients are:

$$H^{1,-1} = Gr^1_F = F^1 = \mathbf{H}^1(0 \longrightarrow \Omega^1_S \otimes \mathcal{F}^0 \longrightarrow \Omega^2_S \otimes Gr^{-1}_{\mathcal{F}}), \tag{2.8}$$
$$H^{0,0} = Gr^0_F = F^0/F^1 = \mathbf{H}^1(\mathcal{F}^0 \longrightarrow \Omega^1_S \otimes Gr^{-1}_{\mathcal{F}}), \tag{2.9}$$
$$H^{-1,1} = Gr^{-1}_F = F^{-1}/F^0 = \mathbf{H}^1(Gr^{-1}_{\mathcal{F}}). \tag{2.10}$$

Considering $H^1(S,\mathbf{V_Q})$ as a lattice of $H^1(S,\mathbf{V_C})$, we set

$$H^1(S,\mathbf{V_Q})^{0,0} = H^1(S,\mathbf{V_Q}) \cap H^{0,0}. \tag{2.23}$$

Let $p_n : H^n(S,\mathbf{V_C}) \longrightarrow H^{-1,n} = H^n(S, Gr^{-1}_{\mathcal{F}})$ be the natural projection map induced by the Hodge spectral sequence. Set also

$$A_{const} = \mathrm{coker}\{p_0 : H^0(S,\mathbf{V_Z}) \to H^0(Gr^{-1}_{\mathcal{F}})\}, \tag{2.11}$$
$$H^1(S,\mathbf{V_Z})^{0,0} = \ker\{p_1 : H^1(S,\mathbf{V_Z}) \to H^1(S, Gr^{-1}_{\mathcal{F}})\}. \tag{2.12}$$

Then by Hodge theory one has

$$H^1(S,\mathbf{V_Q})^{0,0} = H^1(S,\mathbf{V_Z})^{0,0} \otimes_{\mathbf{Z}} \mathbf{Q}. \tag{2.13}$$

Under these notations, we can state the following theorem which gives a very natural description of $MW(A/S)$. (Cf. [Z1, Cor. 10.2].)

(2.14) Theorem. *Assume that S is compact. Then*
(i) *A_{const} in (2.11) is an abelian variety over $\mathbf{C}$, and*
(ii) *we have a natural exact sequence of abelian group*

$$0 \longrightarrow A_{const} \longrightarrow MW(A/S) \longrightarrow H^1(S,\mathbf{V_Z})^{0,0} \longrightarrow 0. \tag{2.15}$$

(2.16) Corollary. *If S is compact, the followings are equivalent.*

(i) $H^0(S, \mathbf{V}_C) = 0$.
(ii) $H^0(S, \mathbf{R}^1 f_* \mathcal{O}_S^{an}) = 0$.
(iii) *$K/\mathbf{C}$-trace A_{const} is zero.*
(iv) *$MW(A/S) \simeq H^1(S, \mathbf{V_Z})^{0,0}$, so it is a finitely generated abelian group.*

If moreover $H^1(S, \mathbf{V_Q})^{0,0} = 0$, then $MW(A/S)$ is a fnite group.

§3 Mordell-Weil groups of Kuga fiber spaces.

Let $G_{\mathbf{Q}}$ be a semisimple $\mathbf{Q}$-algebraic group such that $\mathcal{D} := G_{\mathbf{R}}/K$ becomes a hermitian symmetric domain. A *$\mathbf{Q}$-symmplectic representation* of $G_{\mathbf{Q}}$ is, roughly speaking, a homomorphism $\rho : G_{\mathbf{Q}} \longrightarrow Sp(2g, \mathbf{Q})$ which induces an equivariant holomorphic map

$$h : \mathcal{D} \longrightarrow \mathcal{H}_g = Sp(2g, \mathbf{R})/K'.$$

Pulling back the universal family $\tilde{A}_g \longrightarrow \mathcal{H}_g$ via h, we obtain a family of abelian varieties $A \longrightarrow \mathcal{D}$. Taking a torsion free discrete group $\Gamma \subset \rho^{-1}(Sp(2g, \mathbf{Z}))$, one can obtain an abelian scheme $f : A_\Gamma \longrightarrow S_\Gamma = \Gamma \backslash \mathcal{D}$, which we call *a Kuga fiber space* of abelian varieties associated a symplectic replesentation ρ.

Shioda [Sd1] proved that Mordell-Weil groups of elliptic modular surfaces are finite. Silverberg [Si1, 2 & 3] showed the finiteness of Mordell-Weil groups of Kuga fiber spaces which are characterized by endomorphism algebras and polarizations, introduced by Shimura [Sh1], [Sh2].

By using the result in §2, Borel-Wallach vanishing theorem [B-W] for L^2 cohomology groups and also (Mixed) Hodge theory, the author proved the following

(3.2) Theorem. *(cf. [Sa.MH, 1991]). For a Kuga fiber space associated to a standard $\mathbf{Q}$-symplectic representation, the Mordell-Weil group is finite except possibly for one case.*

On the other hand, Mok and To obtained the following theorem independently.

(3.3) Theorem. *([Mo, 1990], [Mo-T, 1991]). For any Kuga fiber space witha trivial $K/\mathbf{C}$-trace, the Mordell-Weil group is finite.*

Mok announced the above result in [Mo], but in the first version of full paper [Mo-T], there was a misunderstanding about Kuga fiber spaces, that is, they tacitly assumed that the $\mathbf{R}$-valued points $G_{\mathbf{R}}$ has no compact factor, which is not true in many important cases.

§4 Mordell-Weil groups of Elliptic surfaces.

Let $f : A \longrightarrow S$ be an abelian scheme of relative dimension g. In this section we assume that $\dim S = 1$. If S is not compact, in order to have a similar description as in Theorem (2.14), we have to introduce some compactification of both abelian schemes and VHSs. The canonical "compactification" (or extension in precise) of an abelian scheme is given by "Néron model" due to Néron (cf. [A], [B-L-R]), and the canonical extension of local system was given by Deligne [D1], and Zucker [Z1] extend the Hodge theory for this extension.

In order to illustrate this, we will explain about elliptic surfaces. Hence we also assume that g (= the relative dimension of f) is equal to 1. Denote by $\overline{S}$ the compactification of S, and set $\Sigma = \overline{S} - S$. Then we have the following diagram:

$$\begin{array}{ccccc} Y & \hookrightarrow & \overline{A} & \hookleftarrow & A \\ \downarrow & & \downarrow^{\overline{f}} & & \downarrow^{f} \\ \Sigma & \hookrightarrow & \overline{S} & \hookleftarrow_{j} & S. \end{array} \tag{4.1}$$

Here $\overline{A}$ is a smooth projective surface which has no exceptional curve of the first kind in fibers and we set $Y = \overline{A} - A$. The fiber space $\overline{f} : \overline{A} \longrightarrow \overline{S}$ is called an elliptic surface. If we denote by $\overline{A}^{\sharp} \subset \overline{A}$ the smooth part of $\overline{f}$, and by $\overline{A}_0^{\sharp} \subset \overline{A}^{\sharp}$ the connected component in which the zero section is passing. Then $\overline{A}^{\sharp}$ is a smooth commutative group scheme over S which has the Néron's universal property, so we call $\overline{A}^{\sharp}$ *Néron model.* In this case, we have the following isomorphism:

$$MW(A/S) \simeq \{s : \overline{S} \longrightarrow \overline{A}^{\sharp}, \text{a holomorphic section of } \overline{f}\ \}. \tag{4.2}$$

Moreover we define the narrow Mordell-Weil group by

$$MW_0(A/S) \simeq \{s : \overline{S} \longrightarrow \overline{A}_0^{\sharp}, \text{a holomorphic section of } \overline{f}\ \}. \tag{4.3}$$

Setting $\mathbf{V_Z} = \mathbf{R}_1 f_* \mathbf{Z}_A$, we have the following exact sequence due to Kodaira

$$0 \longrightarrow j_*\mathbf{V_Z} \longrightarrow R^1\overline{f}_*\mathcal{O}_A \longrightarrow \mathcal{O}_{\overline{S}}(\overline{A}_0^{\sharp}) \longrightarrow 0. \tag{4.4}$$

Zucker [Z1] showed that $j_*\mathbf{V_Z}$ underlies a cohomological Hodge complex or *a Hodge module* in the sense of Mo. Saito [Sa.Mo1, 2]. In particular, the cohomology group $H^q(\overline{S}, j_*\mathbf{V_Z})$ has a pure Hodge structure of weight $q-1$. Let $\overline{V_{\mathcal{O}}}$ denote a Deligne's quasi-canonical extension of $V_{\mathcal{O}}$. Then the Gauss-Manin connection $\nabla : V_{\mathcal{O}} \rightarrow V_{\mathcal{O}} \otimes \Omega^1_S$ extends to

$$\nabla : \overline{V_{\mathcal{O}}} \rightarrow \overline{V_{\mathcal{O}}} \otimes \Omega^1_{\overline{S}}(log\Sigma). \tag{4.5}$$

Let us set $\overline{\mathcal{F}^p} = j_*\mathcal{F}^p \cap \overline{V_{\mathcal{O}}}$. Then $\overline{\mathcal{F}^p}$ is a locally free extension of $\mathcal{F}^p$ to $\overline{S}$. Then we have the following isomorphism:

$$\overline{Gr_{\mathcal{F}}^{-1}} := \overline{\mathcal{F}^{-1}}/\overline{\mathcal{F}^0} \simeq R^1\overline{f}_*\mathcal{O}_A. \tag{4.6}$$

We set, as in (2.11) and (2.12),

$$A_{const} := H^0(\overline{S}, \overline{Gr_{\mathcal{F}}^{-1}})/H^0(\overline{S}, j_*\mathbf{V_Z}) \tag{4.7}$$

$$H^1(\overline{S}, j_*\mathbf{V_Z})^{0,0} := \ker\{H^1(\overline{S}, j_*\mathbf{V_Z}) \longrightarrow H^1(\overline{S}, \overline{Gr_{\mathcal{F}}^{-1}})\}. \tag{4.8}$$

From Zucker's results, one can see that A_{const} is an abelian variety defined over $\mathbf{C}$ and

$$H^1(\overline{S}, j_*\mathbf{V_Z})^{0,0} \otimes \mathbf{Q} \simeq H^1(\overline{S}, j_*\mathbf{V_Q}) \cap H^{0,0}, \tag{4.9}$$

where $H^{0,0}$ is the Hodge component of type (0, 0) of $H^1(\overline{S}, j_*\mathbf{V_C})$ and

$$H^{0,0} \simeq \mathbf{H}^1(\nabla : \overline{\mathcal{F}^0} \longrightarrow \overline{Gr_{\mathcal{F}}^{-1}} \otimes \Omega^1_{\overline{S}}(log\Sigma)). \tag{4.10}$$

As in Theorem (2.14), we have the following proposition from the exact sequence (4.4).

Proposition (4.11). *Under the above notation, we have the following exact sequence:*

$$0 \longrightarrow A_{const} \longrightarrow MW_0(A/S) \longrightarrow H^1(j_*\mathbf{V_Z})^{0,0} \longrightarrow 0$$

Corollary (4.12). *Under the above notation, the followings are equivalent.*

(i) $H^0(\overline{S}, j_*\mathbf{V}_C) = 0$.
(ii) $H^0(\overline{S}, \mathbf{R}^1\overline{f}_*\mathcal{O}_{\overline{S}^{an}}) = 0$.
(iii) *$K/\mathbf{C}$-trace A_{const} is zero.*
(iv) $MW_0(A/S) \simeq H^1(\overline{S}, j_*\mathbf{V_Z})^{0,0}$, *so it is a finitely generated abelian group.*

If moreover $H^1(\overline{S}, j_*\mathbf{V_Q})^{0,0} = 0$, then $MW_0(A/S)$ and $MW(A/S)$ are finite groups.

Remark (4.13). Since $\overline{f} : \overline{A} \longrightarrow \overline{S}$ is an elliptic surface, if $\overline{f}$ is not trivial, then we have

$$H^0(\overline{S}, \mathbf{R}^1\overline{f}_*\mathcal{O}_{\overline{S}^{an}}) = 0.$$

Hence if $\overline{f}$ is not trivial, we have an isomorphism $MW_0(A/S) \simeq H^1(\overline{S}, j_*\mathbf{V_Z})^{0,0}$. In this case, Shioda [Sd1] showed that the full Mordell-Weil group $MW(A/S)$ is isomorphic to

$$NS(\overline{A})/T$$

where T is a subgroup of the Néron-Severi group $NS(\overline{A})$ generated by the zero section and all components of fibers. Note that we have an isomorphism $H^2(\overline{A}, \mathbf{Z})^{0,0} \simeq NS(\overline{A})$ and the Leray spectral sequence for $\overline{f}$ respects Hodge structure (cf. [Z1]). In our situation, we have a homomorphism ([Sd2], [C-Z])

$$\delta : MW(A/S) \longrightarrow H^1(\overline{S}, j_*\mathbf{V_Z})^{0,0} \otimes \mathbf{Q}$$

and the narrow Mordell-Weil group is given by $\delta^{-1}(H^1(\overline{S}, j_*\mathbf{V_Z})^{0,0})$. On $H^1(\overline{S}, j_*\mathbf{V_Z})^{0,0} \otimes \mathbf{Q}$, one has a bilinear form induced by that on $H^2(\overline{A}, \mathbf{Z})^{0,0}$, and hence we can define a paring on the Mordell-Weil group $MW(A/S)$ which makes $MW(A/S)$ a lattice. Shioda called this the Mordell-Weil lattice of $\overline{A}/\overline{S}$ which has been deeply studied in [Sd 2].

§5 A sketch of a proof of Theorem (3.2).

We will give a sketch of a proof of Theorem (3.2). We may assume that a $\mathbf{Q}$-algebraic group $G_\mathbf{Q}$ is simple and a $\mathbf{Q}$-symplectic representation is primary i.e. sum of irreducible representations which are mutually isomorphic.

Denote by $f : A_\Gamma \longrightarrow S_\Gamma = \Gamma\backslash\mathcal{D}$ a corresponding Kuga fiber space associated to a torsion free discrete subgroup $\Gamma \subset G_\mathbf{Q}$. Set $\mathbf{V_Z} := \mathbf{R}_1 f_* \mathbf{Z}_A$. Assume that $\dim S_\Gamma \geq 2$. In this case, if we can show that

$$H^q(S_\Gamma, \mathbf{V_C}) = 0 \qquad \text{for } q = 0, 1,$$

we obtain the finiteness of the Mordell-Weil group $MW(A/S)$.

Let $\overline{S_\Gamma}$ be the Baily-Borel-Satake compactification of S_Γ. Since codim $(\overline{S_\Gamma}/S_\Gamma) \geq 2$, we have isomorphisms

$$H^q(S_\Gamma, \mathbf{V_C}) \simeq IH^q(\overline{S_\Gamma}, \mathbf{V_C}) \qquad \text{for } q = 0, 1 \ ,$$

where $IH^\cdot(\overline{S_\Gamma}, \mathbf{V_C})$ denote intersection cohomology groups. On the other hand, by Zucker conjecture [L], [Sa-St], one has isomorphisms between intersection cohomology groups and L_2-cohomology groups, which are calculated by some representation theory [B-W]. In fact, by using that, we can show that

$$H^q_{(2)}(S_\Gamma, \mathbf{V_C}) = 0 \text{ for } q < \mathbf{R}\text{-rank of } G_\mathbf{R} \ .$$

Therefore, we have done if $\mathbf{R}$-rank is greater than one. And if $\mathbf{R}$-rank is one, we have the isomorphism

$$G_{\mathbf{R}} \simeq SU(n,1) \times K$$

where K is compact. In this case we can not expect the vanishing of H^1, hence for example in case that S_Γ is compact, we have to use sharper criterion like

$$H^1(S_\Gamma, \mathbf{V_Q})^{0,0} = 0.$$

This is done by a careful study of the $(0,0)$-component and for detail see [Sa.MH]. (The same argument goes through even if $\dim S_\Gamma = 1$ but S_Γ is compact.)

Next, we have to deal with the case when S_Γ is not compact and $\mathbf{R}$-rank of $G_{\mathbf{R}}$ is one, that is, the case when $G_{\mathbf{Q}} = SU(n,1,\mathbf{Q})$.

We will only give a sketch of the proof when $G_{\mathbf{Q}} = SL_2(\mathbf{Q})$ i.e. when $f : A_\Gamma \longrightarrow S_\Gamma$ is an elliptic modular surface. The proofs of other cases are a little bit tricky, but the key idea is the same as in the following.

We only have to show that the narrow Mordell-Weil group $MW_0(A_\Gamma/S_\Gamma)$ is finite. Here we put $A := A_\Gamma$ and $S = S_\Gamma$. Thanks to (4.10) and (4.12), this will be proved if

$$H^{0,0} \simeq \mathbf{H}^1(\overline{\nabla} : \overline{\mathcal{F}^0} \longrightarrow \overline{Gr_{\mathcal{F}}^{-1}} \otimes \Omega^1_{\overline{S}}(log\Sigma)) = 0.$$

Note that the sheaves $\overline{\mathcal{F}^0}$ and $\overline{Gr_{\mathcal{F}}^{-1}} \otimes \Omega^1_{\overline{S}}(log\Sigma)$ are invertible. Since the Gauss-Manin complex over S

$$\nabla : \mathcal{F}^0 \longrightarrow Gr_{\mathcal{F}}^{-1} \otimes \Omega^1_S$$

is induced by a non-trivial homogeneous variation, ∇ must induce an isomorphism. On the other hand, by the uniqueness of the canonical extension, ∇ has to extend to an isomorphism $\overline{\nabla}$, therefore we have

$$\mathbf{H}^1(\overline{\nabla}) = 0$$

as desired.

Remark (5.1). The above argument for elliptic modular surfaces implies that the Hodge decomposition on $H^1(\overline{S}, j_*\mathbf{V_C})$ are given by

$$H^1(\overline{S}, j_*\mathbf{V_C}) = H^{1,-1} \oplus H^{-1,1}.$$

The space $H^{1,-1}$ is isomorphic to the space of holomorphic cusp forms. This is the easiest case of Eichler-Shimura isomorphism which was reformulated by Zucker [Z1] in this form.

§6 Néron models of Jacobians of curves over function fields.

In this section, we will discuss Néron models of Jacobians of curves over function fields. A typical examples are given by elliptic surfaces as in §4, but we will deal with curves with arbitrary genus.

As before, let S be a connected smooth curve, $\overline{S}$ its smooth compactification, and set $\Sigma = \overline{S} - S$. We denote by K the function field $\mathbf{C}(S)$ of S (or $\overline{S}$).

Consider a projective smooth morphism $f : X \longrightarrow S$ whose geometric fibers are smooth connected curves with genus $g \geq 1$. Moreover we always consider the following diagram;

$$
(6.1) \qquad \begin{array}{ccccc}
Y & \hookrightarrow & \overline{X} & \hookleftarrow & X \\
\downarrow & & \downarrow^{\overline{f}} & & \downarrow^{f} \\
\Sigma & \hookrightarrow & \overline{S} & \hookleftarrow_{j} & S.
\end{array}
$$

Here $\overline{X}$ is a smooth projective surface without exceptional curves of the first kind in fibers of $\overline{f}$, $\overline{f}$ is a compactification of f, which is a projective flat morphism. The generic fiber X_K of $f : X \longrightarrow S$ (or $\overline{f} : \overline{X} \longrightarrow \overline{S}$) is a proper smooth curve defined over the field K. The Jacobian of curve X_K is defined to be

$$J_{X_K} := Pic^0_{X_K/K}$$

which is an abelian variety of dimension g over K. Let $Pic_{X/S}$ (resp. $Pic_{\overline{X}/\overline{S}}$) denote the relative Picard functor for f (resp. $\overline{f}$) (cf. [B-L-R, 8.1]). Then since $f : X \longrightarrow S$ is a projetive smooth morphism, the functor $Pic_{X/S}$ is represented by a smooth separated S-scheme which is also denoted by $Pic_{X/S}$ (cf. [B-L-R, 9-3]).

Moreover one has a decomposition

$$Pic_{X/S} = \coprod_{n \in \mathbf{Z}} Pic^n_{X/S}$$

where $Pic^n_{X/S}$ denote the open and closed subscheme of $Pic_{X/S}$ consisting of all line bundles of degree n. The subscheme $Pic^0_{X/S}$ becomes an abelian scheme over S, which is denoted by $J_{X/S}$ ([B-L-R, 9-4]), and moreover S has a canonical S-ample rigidified line bundle L on J. The abelian scheme $\pi : J_{X/S} \longrightarrow S$ is the Néron model of J_{X_K}/K over S ([B-L-R, 9.5, Th.1]). In particular, we have a canonical isomorhpism

$$MW(J_{X_K}/K) = J_{X_K}(K) \xrightarrow{\sim} J_{X/S}(S)$$

where $J_{X/S}(S)$ denote the group of regular sections of f.

In order to obtain the Néron model over $\overline{S}$, we have to extend the abelian scheme $\pi : J_{X/S} \longrightarrow S$ to some group scheme over $\overline{S}$.

For each $s \in \Sigma$, let $X_s = \Sigma_{i=1}^{l} m_i X_i$ denote the scheme theoretic fiber of s with the decomposition into irreducible components. We assume that for all $s \in \Sigma$, $g.c.d(m_i) = 1$. This condition is satisfied if e.g. $f : X \longrightarrow S$ has a section. In this case by using a result due to Raynaud theorem (cf. [B-L-R, 9.4, Th. 2]) $Pic_{\overline{X}/\overline{S}}$ is an algebraic space over $\overline{S}$ and $Pic^0_{\overline{X}/\overline{S}}$ is a separated $\overline{S}-$ scheme. Consder a subfunctor $\overline{P}$ of $Pic_{\overline{X}/\overline{S}}$ which is defined as the kernel of the degree morphism $deg : Pic_{\overline{X}/\overline{S}} \to \mathbf{Z}$. Then $\overline{P}$ can be locally considered as a scheme theoretic closure of $Pic^0_{X/S}$ in $Pic_{\overline{X}/\overline{S}}$. Moreover let $\overline{E}$ denote the scheme theoretic closure of zero section $\epsilon : S \longrightarrow Pic^0_{X/S}$ in $\overline{P}$.

Now we can state the fundamental result.

Theorem6.2. *([B-L-R, 9.5, Th.4]) Under the above notations and assumptions, we have the following:*

(i) *The quotient $\overline{J} = \overline{P}/\overline{E}$ exists as a separated $\overline{S}$-group scheme and is the Néron model of J_{X_K} over $\overline{S}$.*

(ii) *$Pic^0_{\overline{X}/\overline{S}}$ is a separated $\overline{S}$-scheme and coincides with the identity component $\overline{J}^0$ of the Néron model $\overline{J}$ of J_{X_K}.*

The group of connected components of Néron model.

In [B-L-R, 9.6], they calculated the group of connected component of the singular fiber of Néron model $\overline{J}/\overline{J}^0$by using the intersection number of the singular fiber X_s . We will show that there exists another approach by using the monodoromy on a nearby fiber. This approach seems to be very hopeful for general abelian scheme which is not neccesarily a Jacobian.

Under the same notations and assumptions as in Theorem 6.2, we have the following exact sequence of the sheaf on S^{an}.

$$0 \longrightarrow \mathbf{R}^1 f_* \mathbf{Z} \longrightarrow R^1 f_* \mathcal{O}_X^{an} \longrightarrow \mathcal{O}_S^{an}(J_{X/S}) \longrightarrow 0. \tag{6.3}$$

It is easy to show that the identity component $\overline{J}^0$ of the Néron model $\overline{J}$ fits into the following exact sequence on $\overline{S}^{an}$:

$$0 \longrightarrow j_* \mathbf{R}^1 f_* \mathbf{Z} \longrightarrow R^1 \overline{f}_* \mathcal{O}_X^{an} \longrightarrow \mathcal{O}_S^{an}(\overline{J}^0_{X/S}) \longrightarrow 0. \tag{6.4}$$

Moreover one can define $Tor = \oplus_{s \in \Sigma} Tor_s$ by

$$0 \longrightarrow \overline{J}^0_{X/S} \longrightarrow \overline{J}_{X/S} \longrightarrow Tor \longrightarrow 0. \tag{6.5}$$

Let Δ denote the small nbd of a critical value $s \in \Sigma$ with coordinate t. Fix a point $t \in \Delta - \{0\}$ and consider the monodromy transformation $T = T_s : H^1(X_t, \mathbf{Z}) \to H^1(X_t, \mathbf{Z})$. Note that $H^1(X_t, \mathbf{Z})$ is a free $\mathbf{Z}$-module of rank $2g$. Consider the following homomorphism

$$N := T - I_{2g} : H^1(X_t, \mathbf{Z}) \longrightarrow H^1(X_t, \mathbf{Z}).$$

Then one has isomorphisms

$$\text{(6.6)} \qquad \mathrm{Ker} N \simeq (j_* R^1 f_* \mathbf{Z}_X)_s, \qquad \mathrm{Coker} N \simeq (R^1 j_* (R^1 f_* \mathbf{Z}_X))_s.$$

From the above sequence, one has the following theorem.

Theorem 6.7. *Under the above notations and assumptions, we have an isomorphism*

$$\text{(6.8)} \qquad Tor_s \simeq \text{Torsion part of Coker} N$$

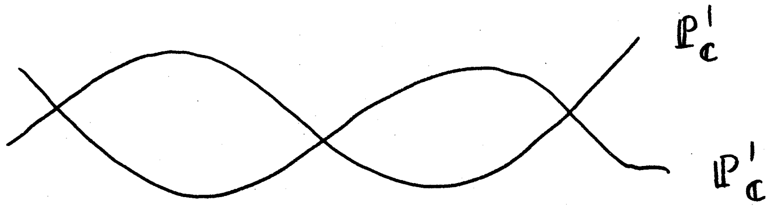
Remark 6.9. A degenerate elliptic curve of type $\mathbf{I}_b^*$ has a local monodromy

$$T = \begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix},$$

while the grroup of connected compnents is one of

$$\mathbf{Z}/2 \oplus \mathbf{Z}/2, \quad \text{or} \quad \mathbf{Z}/4$$

depending on the parity of $-b$. Ueno and Namikawa classified all degenerate curves of genus 2 with explicit equations and local monodromy. One can try to calculate the Tor for the stable curve of genus 2



with the local monodromy

$$T = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\Rightarrow \quad Tor_s \cong \mathbb{Z}/3\mathbb{Z}.$$

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