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ON THE DIFFERENTIAL OPERATOR $D^n f(z)$

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1. Introduction

Let A be the class of analytic functions $f(z)=z+a_2z^2+\cdots\cdots$ in the unit disk U $=\{z; |z|<1\}$. We define the differential operator $D^n f(z)$ for $f(z)\in A$, according to Salagean [7], by

(1.1) $$D^0f(z)=f(z),\ D^1f(z)=Df(z)=zf'(z),\quad D^nf(z)=D(D^{n-1}f(z)), n\geqq 2.$$

Then we can easily see from the definition that

(1.2) $$D^nf(z)=z+2^na_2z^2+3^na_3z^3+\cdots\cdots, \text{ and } f(z)\in A,$$

(1.3) $$\frac{D^1f(z)}{D^0f(z)}=\frac{zf'(z)}{f(z)},\quad \frac{D^2f(z)}{D^1f(z)}=\frac{z(zf'(z))'}{zf'(z)}=1+\frac{zf''(z)}{f'(z)},$$

(1.4) $$\frac{D^{n+2}f(z)}{D^{n+1}f(z)}=\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}=1+\frac{z(D^nf(z))''}{(D^nf(z))'}.$$

We defined the class O_n of functions $f(z)\in A$ which ,was first done by Obradović, to satisfy

(1.5) $$\mathrm{Re}\{\frac{D^{n+1}f(z)}{D^nf(z)}\}<\frac{n+2}{n+1},\quad z\in U.$$

We showed the sufficient conditions of f(z) to be in O_n ealier in [1].

In this paper we study the class $F_n(\alpha)$ of functions $f(z)\in A$ which satisfy

(1.6) $$\mathrm{Re}\{\frac{D^{n+1}f(z)}{D^nf(z)}\}<\alpha,\quad z\in U,$$

where α is some real number and $\alpha>1$.

We will use well-known symbols and terms without definitions.

2. Main theorem of $F_n(\alpha)$

First, we have the following Theorem 1 by the definition of $D^n f(z)$ with the aid of Lemma 1 (Jack [2]).

LEMMA 1. Let $w(z)$ be a non-constant and analytic function in U with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point z_0, we have $z_0 w'(z_0)=m\, w(z_0)$, where m is a real number and $m\geqq 1$.

THEOREM 1. For some real number $\alpha>1$ and for any natural number n if the condition

$$(2.1)\qquad \mathrm{Re}\left\{\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right\}<\alpha,\quad z\in U$$

is satisfied, then the following inequality

$$(2.2)\qquad \mathrm{Re}\left\{\frac{D^{n+1}f(z)}{D^{n}f(z)}\right\}<\beta,\quad z\in U$$

holds, where

$$(2.3)\qquad \beta=\frac{2\alpha-1+\sqrt{(4\alpha^2-4\alpha+9)}}{4}.$$

Proof of Theorem 1. We put $p(z)=\dfrac{D^{n+1}f(z)}{D^n f(z)}$, then it holds that $p(z)=\dfrac{z(D^n f(z))'}{D^n f(z)}$,

and

$$(2.4)\qquad \frac{D^{n+2}f(z)}{D^{n+1}f(z)}=1+\frac{z(D^n f(z))'}{(D^n f(z))'}=p(z)+\frac{zp'(z)}{p(z)}.$$

The function $p(z)$ is analytic and $p(z)\neq 0$ in U from the hypothesis (2.1) and $p(0)=1$.

We define a function $w(z)$ by $p(z)=\dfrac{1-(2\beta-1)w(z)}{1-w(z)}=\dfrac{1-\lambda w(z)}{1-w(z)}$, where we put $\lambda=2\beta-1$, $\beta>1$. We can see that $w(z)$ is analytic in U and $w(z)\neq 1$. Moreover, it holds that

$$(2.5)\qquad \begin{aligned}&\mathrm{Re}\{p(z)\}<\beta \iff |w(z)|<1 \text{ for any } z\in U, \text{ and}\\ &\mathrm{Re}\{p(z_0)\}=\beta \iff |w(z_0)|=1.\end{aligned}$$

Putting $Q(z)=p(z)+\dfrac{zp'(z)}{p(z)}$, $Q(z)$ is analytic in U and $Q(0)=1$. So we have

$$(2.6)\qquad Q(z)=\frac{1-\lambda w(z)}{1-w(z)}-\frac{\lambda zw'(z)}{1-\lambda w(z)}+\frac{zw'(z)}{1-w(z)}$$

$$=\frac{1-\lambda w(z)}{1-w(z)}+\frac{zw'(z)}{w(z)}\{\frac{-\lambda w(z)}{1-\lambda w(z)}+\frac{w(z)}{1-w(z)}\}.$$

Now we assume that $|w(z)|<1$ for $|z|<|z_0|<1$ and $|w(z_0)|=1$ for $|z|=|z_0|$, then we have $\frac{z_0w'(z_0)}{w(z_0)}=m\geqq 1$ by Lemma 1. We put $w(z_0)=e^{i\theta}$ and we have

$Q(z_0)=\frac{1-\lambda e^{i\theta}}{1-e^{i\theta}}+m\{\frac{-\lambda e^{i\theta}}{1-\lambda e^{i\theta}}+\frac{e^{i\theta}}{1-e^{i\theta}}\}$, hence it holds the inequalities

$\mathrm{Re}\{Q(z_0)\}\geqq\frac{1+\lambda}{2}+m\{\frac{\lambda-1}{2(1+\lambda)}\}\geqq\frac{(2\beta-1)(\beta+1)}{2\beta}=\alpha$. This fact contradicts to the condition (2.1), so it must be $|w(z)|<1$, $z\in U$. This implies that $\mathrm{Re}\{\frac{D^{n+1}f(z)}{D^nf(z)}\}=\mathrm{Re}\{p(z)\}<\beta$, $z\in U$ and this completes the proof.

When $\alpha>1$ and $\beta=\frac{2\alpha-1+\sqrt{(4\alpha^2-4\alpha+9)}}{4}$, we have $\alpha>\beta>1$. So we can express Theorem 1 about the relations between $F_n(\alpha)$ as follows:

Theorem 1'. For every $\alpha>1$, it holds that $F_{n+1}(\alpha)\subset F_n(\beta)$, $n=1,2,3,\cdots\cdots$. Where β is the same value of (2.3).

3. Marx-Strohhäcker differential subordinate systems

Miller and Mocanu have proved interesting theorems in [4] about Marx-Strohhäcker differential subordinate systems of the first type and second type. Their theorems can be rewritten as the following two lemmas. Subordination is denoted by $f(z)\prec g(z)$.

LEMMA 2. For $f(z)\in A$ and $k(z)\in A$ if the condition

$$(3.1)\qquad \frac{zf''(z)}{f'(z)}\prec\frac{zk''(z)}{k'(z)}$$

is satisfied then we have the following results, (3.2) $$\frac{zf'(z)}{f(z)} \text{ is analytic in } U \text{ and } \frac{zf'(z)}{f(z)} \prec \frac{zk'(z)}{k(z)},$$

where k(z) must satisfy (3.3), (3.4) and (3.5).

(3.3) $$q(z)=\frac{zk'(z)}{k(z)} \text{ is univalent in } U,$$

(3.4) $$\mathrm{Re}\{1+\frac{zq'(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\}>0,\ z\in U,$$

(3.5) $$\mathrm{Re}\{q(z)+1+\frac{zq'(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\}>0,\ z\in U.$$

LEMMA 3. For $f(z)\in A$ and $k(z)\in A$ it holds that

(3.6) $$\frac{zf'(z)}{f(z)} \prec \frac{zk'(z)}{k(z)} \Rightarrow \frac{f(z)}{z} \prec \frac{k(z)}{z}.$$

Where k(z) must satisfy the following two conditions,

(3.7) $$\frac{zk'(z)}{k(z)}-1 \text{ is starlike with respect to the origin in } U, \text{ and}$$

(3.8) $$\frac{k(z)}{z} \text{ is univalent in } U.$$

Lemma 2 and Lemma 3 in the paper are said by Miller and Mocanu to be Marx-Strohhäcker differential subordinate systems of the first type and second type, respectively. We have rewritten their theorem 2 [4] as Lemma 2. Lemma 3 is the direct result of their theorem 4 [4] by putting $p(z)=f(z)/z$, and their Corollary 4.2 [4] shows that the conditions (3.7) and (3.8) of Lemma 3 are replaceable by (3.7)' and (3.8)' as a special case.

(3.7)' $$\mathrm{Re}\{\frac{zk'(z)}{k(z)}\}>0 \text{ for any } z\in U, \text{ and}$$

(3.8)' $$\frac{zk'(z)}{k(z)} \text{ is convex in } U.$$

We will prove the following Theorem 2 by using lemma 2 and lemma 3.

THEOREM 2. For $f(z) \in A$ if it holds that

$$(3.9) \qquad \mathrm{Re}\{1+\frac{zf''(z)}{f'(z)}\}<\frac{3}{2}, \quad z \in U,$$

then we have the following results,

$$(3.10) \qquad \frac{zf'(z)}{f(z)} \prec \frac{2(1-z)}{2-z} \text{ and } f(z) \text{ is bounded in } U.$$

Conversely, if it holds that $\frac{zf'(z)}{f(z)} \prec \frac{2(1-z)}{2-z}$, then $f(z)$ satisfies (3.9).

Proof. Suppose it holds (3.9) for $f(z) \in A$, then there exists the function $k(z) = z - \frac{1}{2}z^2$ which satisfies $1+\frac{zk''(z)}{k'(z)} = \frac{1-2z}{1-z}$. Since the function $\frac{1-2z}{1-z}$ maps U onto the half plane $\{ w ; \mathrm{Re}\{w\}<3/2 \}$, $k(z)$ satisfies evry conditions of Lemma 2. Hence we have $\frac{zf'(z)}{f(z)} \prec \frac{zk'(z)}{k(z)} = \frac{2(1-z)}{2-z}$ and $\left| \frac{zk'(z)}{k(z)} - \frac{2}{3} \right| < \frac{2}{3}$, $z \in U$. Therefore $f(z)$ is starlike in U. On the other hand, since the function $k(z)$ satisfies the conditions (3.7) and (3.8) of Lemma 3, we have $\frac{f(z)}{z} \prec \frac{k(z)}{z} = 1 - \frac{z}{2}$ and $|f(z) - z| < \frac{1}{2}|z|^2$. Therefore $f(z)$ is bounded in U.

Conversely, if we assume $\frac{zf'(z)}{f(z)} \prec \frac{2(1-z)}{2-z}$, then we can put $\frac{zf'(z)}{f(z)} = \frac{2(1-w(z))}{2-w(z)}$, where $w(z)$ is, a so called Schwarzian function, analytic in U and $w(0)=0$, $|w(z)| < 1$ for every $z \in U$. After a simple calculation, we have the inequality (3.9) by virtue of Lemma 1.

The following Theorem 2′ which is an extension of Theorem 2 will be proven in another paper, because we do not need here the result of Theorem 2′ in the below discussion and the proof of Theorem 2′ is not so simple.

THEOREM 2′. For $f(z) \in A$ if $f(z)$ satisfies $1 < \alpha \leqq 3/2$

$$(3.11) \qquad \mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} < \alpha, \quad z \in U,$$

then we have

$$(3.12) \qquad \frac{zf'(z)}{f(z)} \prec \frac{(2\alpha-1)z(1-z)^{2(\alpha-1)}}{1-(1-z)^{2(\alpha-1)}},$$

and $f(z)$ is bound in U, Conversely, if it satifies (3.12) then (3.11) holds. Where $\alpha = 3/2$.

There is some history to (3.9) for $f(z) \in A$. If we assume (3.9), then $f(z)$ is univalent in U was shown by Ozaki in 1941 [5]. Later in 1952, Umezawa showed that $f(z)$ is convex in one direction [9], and in 1973 Sakaguchi showed that $f(z)$ is close-to-convex [6]. In 1982 R.Singh and S.Singh [8] showed that if $f(z)$ satisfies the condition (3.9), then (3.10) holds. Here we have used another way to show the same result (3.10) and we showed that the converse is true, too.

We have the following analogous theorem of Theorem 2′. But the proof of our Theorem 3 is only to check that $k(z)$ satisfies every condition of Lemma 3. So we omit it.

THEOREM 3. For $f(z) \in A$ if it holds that $1 < \alpha \leqq 2$

$$(3.3) \qquad \mathrm{Re}\{\frac{zf'(z)}{f(z)}\} < \alpha, \; z \in U,$$

then we have

$$(3.4) \qquad \frac{f(z)}{z} \prec (1-z)^{2(\alpha-1)},$$

and $f(z)$ is bounded in U. Conversely, if it satisfies (3.4) then (3.13) holds. Where $\alpha = 2$ is the best possible value.

4. On the special case of $F_n(\alpha)$

In this section the following lemma (Miller and Mocanu [3, theorem 10(ii)]) is needed in order to prove our Theorem 4.

LEMMA 4. Let N(z) and D(z) be regular in U with N(0)=D(0)=0, and let γ be real. If D(z) maps U onto a (possibly many-sheeted) region which is starlike with respect to the origin, then we have

$$(4.1)\qquad \mathrm{Re}\{\frac{N'(z)}{D'(z)}\}<\gamma,\ z\in U \Longrightarrow \mathrm{Re}\{\frac{N(z)}{D(z)}\}<\gamma,\ z\in U.$$

THEOREM 4. It holds that $F_{n+1}(\alpha) \subset F_n(\alpha)$ for $1<\alpha \leqq 3/2$, $n=1,2,3,\cdots\cdots$.

Proof. For $f(z)\in F_{n+1}(\alpha)$, we have $\frac{D^{n+2}f(z)}{D^{n+1}f(z)}=\frac{z(D^{n+1}f(z))'}{z(D^n f(z))'}=\frac{(D^{n+1}f(z))'}{(D^n f(z))'}$ from the definition of $D^n f(z)$. Since it holds $\mathrm{Re}\{\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\}=\mathrm{Re}\{1+\frac{z(D^n f(z))'}{(D^n f(z))'}\}<\alpha \leqq 3/2$, it follows that $D^n f(z)$ is a starlike function in U from Theorem 2. Putting $N(z)=D^{n+1}f(z)$ and $D(z)=D^n f(z)$, we have $\mathrm{Re}\{\frac{D^{n+1}f(z)}{D^n f(z)}\}<\alpha$, $z\in U$ by Lemma 4. Therefore, $f(z)\in F_n(\alpha)$.

When we put $\alpha = 3/2$ in Theorem 1', we obtained

$$(4.2)\qquad F_{n+1}(\frac{3}{2})\subset F_n(\frac{1+\sqrt{3}}{2}),\ n=1,2,3,\cdots\cdot,\ \frac{1+\sqrt{3}}{2}=1.361\cdots\cdot.$$

On the other hand, if $f(z)\in F_{n+1}(\frac{3}{2})$ then we have $\mathrm{Re}\{1+\frac{z(D^n f(z))'}{(D^n f(z))'}\}<\frac{3}{2}$ and $\frac{z(D^n f(z))'}{D^n f(z)}\prec\frac{2(1-z)}{2-z}$ from Theorem 2. This implies that $\mathrm{Re}\{\frac{D^{n+1}f(z)}{D^n f(z)}\}<\frac{4}{3}$, $z\in U$, hence $f(z)\in F_n(\frac{4}{3})$. Since $F_{n+1}(\frac{3}{2})\subset F_n(\frac{4}{3})\subset F_n(\frac{1+\sqrt{3}}{2})$, this result is more sharp include relation than Theorem 1'.

If $f(z) \in F_1(\alpha)$, $1 < \alpha \leqq \frac{3}{2}$, then it holds $\mathrm{Re}\{\frac{D^2f(z)}{D^1f(z)}\}=\mathrm{Re}\{1+\frac{zf''(z)}{f'(z)}\}$ $< \alpha \leqq \frac{3}{2}$ and $F_1(\alpha) \subset F_1(\frac{3}{2})$. Hence the conclusion of Theorem 2 is reached. This implies that $0 < \mathrm{Re}\{\frac{zf'(z)}{f(z)}\} < \frac{3}{4}$, especially $f(z) \in F_0(\frac{4}{3})$.

If $f(z) \in F_0(\alpha)$, $1 < \alpha \leqq 2$, then it holds $\mathrm{Re}\{\frac{D^1f(z)}{D^0f(z)}\}= \mathrm{Re}\{\frac{zf'(z)}{f(z)}\}< \alpha$ $\leqq 2$ and $F_0(\alpha) \subset F_0(2)$. Similar to the above, the conclusion of Theorem 3 is reached. Particularly, in a case of $\alpha = \frac{3}{2}$ we have that $\frac{f(z)}{z} \prec (1-z)^2$ and $f(z)$ is bounded in U. This fact is stated as follow:

THEOREM 5. (i) If $f(z) \in F_1(\alpha)$, $1 < \alpha \leqq \frac{3}{2}$, then $f(z) \in F_0(\frac{4}{3})$ and the result of Theorem 2, (3.10) is satisfied.

(ii) If $f(z) \in F_0(\alpha)$, $1 < \alpha \leqq 2$, then the result of Theorem 3 is satisfied.

Also, by virtue of Theorem 4 and Theorem 5 Theorem 6 follows.

THEOREM 6. If $f(z) \in F_n(\alpha)$, $1 < \alpha \leqq \frac{3}{2}$, $n=1,2,3,\cdots$, then $f(z) \in F_1(\frac{4}{3})$ $f(z) \in F_0(\frac{4}{3})$.

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