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On the Discrete Boltzmann Equation with Linear and Nonlinear Terms

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Chapter 1
Formulation and results

The discrete model of Boltzmann equations system in a thin infinite tube as follows :

(B)
$$\begin{cases} \dfrac{\partial u_i}{\partial t} + c_i \dfrac{\partial u_i}{\partial x} = Q_i(u) + L_i(u) \\ u_i(x,0) = u_i^0(x) \quad \text{for} \quad x \in \mathbf{R}, t \in \mathbf{R}_+ \end{cases}$$

where

$$Q_i(u) = \sum_{j,k,\ell \in I} (A_{ij}^{k\ell} u_k u_\ell - A_{k\ell}^{ij} u_i u_j)$$

$$L_i(u) = \sum_{k \in I} (\alpha_i^k u_k - \alpha_k^i u_i).$$

The natural physical conditions are following :

Condition 1.—

$$A_{ij}^{k\ell} \geqq 0, \quad A_{ij}^{k\ell} = A_{ji}^{k\ell} = A_{ij}^{\ell k}$$

$$A_{k\ell}^{ij} \neq 0 \quad \Rightarrow \quad i \neq j \quad \text{and} \quad c_i + c_j = c_k + c_\ell$$

$$\forall i \ \exists (j,k,\ell) \quad \text{such that} \quad A_{k\ell}^{ij} \neq 0$$

$$\alpha_i^k \geqq 0$$

Condition 2.—

$$\forall i \in I \ , \ \sum_{k \in I} \alpha_k^i (c_k - c_i) = 0$$

We put I_k, $k = 0, 1, \cdots$ as follows :

$$\begin{aligned} I_0 &= \{i; \alpha^i_k = 0 \text{ for all } k \in I\} \\ &= \{i; \text{particles with velocity } c_i \text{ don't provoke any reflection}\} \\ I_1 &= \{i \notin I_0; \text{there exists } j \in I_0 \text{ such that } \alpha^i_j > 0\} \\ &= \{i; \text{particles with velocity } c_i \text{ is transformed} \\ &\qquad \text{into a particle with velocity } c_j, j \in I_0 \text{ by reflection } \}, \\ I_{k+1} &= \{i \notin \bigcup_{\ell=0}^{k} I_\ell; \text{ there exists } j \in I_k \text{ such that } \alpha^i_j > 0\} \end{aligned} \tag{1.1}$$

Remark : As we see later, I_0 is not empty, if we assume Condition 2.

Proposition 1.1.— *Suppose Conditions 1 and 2. Let $u_i = u_i(x,t) \in C^1(\mathbf{R}_+, \mathcal{S}(\mathbf{R}))$ $(i \in I)$ a solution of (B). Then, for any $t \in \mathbf{R}_+$, we have*

$$\int_{\mathbf{R}} \sum_i u_i(x,t)dx = \int_{\mathbf{R}} \sum_i u_i^0(x)dx \equiv \mu \tag{1.2}$$

(mass conservation law)

$$\int_{\mathbf{R}} \sum_i c_i u_i(x,t)dx = \int_{\mathbf{R}} \sum_i c_i u_i^0(x)dx \tag{1.3}$$

(momentum conservation low)

Proposition 1.2.— *Condition 2 implies that I_0 is not empty.*

The results of this note are following :
<u>under Condition 2</u>

Theorem 1.— *Suppose Conditions 1 and 2. For a Cauchy data u_i^0 positive, summable and bounded, there exists a unique global bounded solution $u_i(x,t) \in L^\infty(\mathbf{R} \times \mathbf{R}_+)$ and*

$$u_i(x,t) \leqq (1 + \sup_{i,x} u_i^0(x)) \exp{(a\mu^2 + b\mu)} \tag{1.4}$$

where a and b depend only on the equations system, and μ is the total mass.

Corollary 2.— *Suppose Conditions 1 and 2. For a Cauchy data u_i^0 positive and bounded, there exists a unique global solution $u_i(x,t) \in L^\infty_{\ell oc}(\mathbf{R} \times \mathbf{R}_+)$ and*

$$u_i(x,t) \leqq \exp{(A\mu^2 t^2 + B)} \tag{1.5}$$

where A and B don't depend on time.

Theorem 3.— *Suppose hypotheses like as in Theorem 1. We have a asymptotical behavior of a solution : there is a function $v_i^T(x,t)$ which verifies that, for any $\varepsilon > 0$, there exists a large T such that for $t > T$*

$$\left\|u_i(\cdot,t) - v_i^T(\cdot,t)\right\|_{L^p} < \varepsilon \ (1 \leqq p \leqq \infty) \quad \text{for all} \quad i \tag{1.6}$$

and

$$\begin{aligned}&\left\|v_i^T(\cdot,t)\right\|_{L^p}\\ &\leqq \begin{cases} m(t) & i \in I_0 \\ m(t)\exp\{-\lambda t(1-\frac{1}{p})\} & i \notin I_0 \end{cases}\end{aligned} \tag{1.7}$$

for all i, where $2 \leqq p \leqq \infty$ and $m(t)$ is a strictly decreasing function.

without Condition 2 We put other hypotheses.

Condition 3.—

$$\alpha_i^k > 0 \quad \text{for} \quad i \neq k$$

Proposition 1.3.— *Condition 2 is not compatible with Condition 3.*

Moreover we assume the microreversibility of the reflections :

Condition 4.—

$$\alpha_i^k = \alpha_k^i$$

In this situation, we can apply the theory due to Shizuta and Kawashima [9] for a positive, summable and bounded Cauchy data and obtain the following theorem :

Theorem 4.— *Assume Conditions 1, 3 and 4. Then, for a smooth, positive, summable and bounded Cauchy data u_i^0 such that $\hat{u}_i^0$ is defined and in L^1, we have a decay estimate for the solution:*

$$\|u_i(\cdot,t)\|_{L^\infty} \leqq C_*(1+t)^{-1/2}(\left\|u^0\right\|_{L^1} + \left\|\hat{u}^0\right\|_{L^1}) \tag{1.8}$$

where the constant C_* depends only on the equations system.

for small Cauchy data
i)Case with the binary collision terms
In this case, we treat general form of the binary collision terms :

$$Q_i(u) = \sum_{jk} B_i^{jk} u_j u_k, \tag{gQ}$$

which is introduced by Bony [4]. In 1990, he showed that the global existence of the solution for small Cauchy data in the case of $L_i = 0$ in $\mathbf{R}^N$ and defined the corresponding wave and scattering operators.

The equations system is following :

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + c_i \dfrac{\partial u_i}{\partial x} = Q_i(u) + L_i(u) \\ u_i|_{t=0} = u_i^0(\cdot) \end{cases} \tag{B}$$

with L_i is of form as before. On this system, we impose some assumptions :

Condition 5.—

$$B_i^{jk} \neq 0 \quad \Rightarrow \quad j \neq k$$
$$B_i^{jk} \neq 0 \quad \Rightarrow \quad j \text{ and } k \notin I_0$$
$$\alpha_i^k \geqq 0$$

Condition 6.—

$$\begin{cases} I_0 \neq \emptyset \\ i \in I \backslash I_0 \Longrightarrow i \in I_1 \end{cases}$$

Remark : Condition 6 means that the particles which don't provoke any reflection don't make any binary collision.

Theorem 5.— *Suppose Conditions 5 and 6. If the Cauchy data is sufficiently small in $H^s(s = 1, 2, \cdots)$, the solution has a decay estimate as follows :*

$$\|u_i\|_{H^s}(\text{ so } \|u_i\|_{L^\infty}) \leqq \begin{cases} C_* \|u^0\|_{H^s} & \text{for } i \in I_0 \\ C_* \|u^0\|_{H^s} e^{-\frac{1}{2}\lambda t} & \text{for } i \in I_1 \end{cases} \tag{1.9}$$

where C_* depends only on the equations system and constant $\lambda > 0$.

ii)Case with the multiple collision terms
The case with the multiple collision terms is studied only in a few papers [1][2][6]. We consider the general multiple collision terms as follows :

$$R_i(u) = \sum_{p=2}^{\sigma} \sum_{j_1} \cdots \sum_{j_p} E_i^{j_1 \cdots j_p} u_{j_1} \cdots u_{j_p} \tag{R}$$

where we permit the cases $j_k = j_\ell, k \neq \ell$.
Then the equations system is following :

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + c_i \dfrac{\partial u_i}{\partial x} = R_i(u) + L_i(u) \\ u_i|_{t=0} = u_i^0(\cdot) \end{cases} \tag{M}$$

where L_i is of form as before. On this system, we impose some assumptions :

Condition 7.—

$$E_i^{j_1 \cdots j_p} \neq 0 \quad \Rightarrow \quad \exists j_\alpha \neq j_\beta, j_\alpha, j_\beta \in \{j_1, \cdots j_p\}$$
$$E_i^{j_1 \cdots j_p} \neq 0 \quad \Rightarrow \quad \begin{cases} \exists j_\alpha \notin I_0 & \text{if } i \in I_0 \\ \exists j_\alpha \neq j_\beta \notin I_0 & \text{if } i \notin I_0 \end{cases}$$
$$\alpha_i^k \geqq 0$$

The similar result is then obtained :

Theorem 6.— *Suppose Conditions 6 and 7. If the Cauchy data is sufficiently small in $H^s(s = 1, 2, \cdots)$, the solution has a decay estimate as follows :*

$$\|u_i\|_{H^s}(\text{ so } \|u_i\|_{L^\infty}) \leqq \begin{cases} C_* \|u^0\|_{H^s} & \text{for} \quad i \in I_0 \\ C_* \|u^0\|_{H^s} e^{-\frac{1}{2}\lambda t} & \text{for} \quad i \in I_1 \end{cases} \tag{1.10}$$

where C_* depends only on the equations system and constant $\lambda > 0$.

Chapter 2
On the proof

§2.1 Estimations

In this section, assuming Conditions 1. and 2, we establish estimations of solutions, improving the method due to Bony [3]. We assume, for simplicity, that $c_i \neq c_j$ for $i \neq j$. This hypothesis is not essential at all and we can recover it by the usual argument.

Let's define Bony's function [3] and its variation :

$$\varphi(t) = \sum_{i,j}(c_i - c_j) \iint \text{sgn}(y - x) u_i(x, t) u_j(y, t) dx dy \tag{2.1}$$

$$\psi(t; x_0, c_0) = \sum_i (c_i - c_0) \int \text{sgn}\{x - (x_0 + c_0 t)\} u_i(x, t) dx \tag{2.2}$$

Differentiating these functions, we have

Lemma 2.1.— *Suppose $T < T^*$. Under Conditions 1 and 2, we have*

$$\Delta(0, T) \leqq C\mu^2 \tag{2.3}$$

$$\delta(0, T) \leqq C\mu \tag{2.4}$$

where T^* is the existence time of solutions and

$$\Delta(t_1, t_2) = \sup_{c_i \neq c_j} \int_{t_1}^{t_2} \int_{\mathbf{R}} u_i(x, t) u_j(x, t) dx dt \tag{2.5}$$

$$\delta(t_1,t_2) = \sup_{c_i \neq c_j} \sup_{x \in \mathbf{R}} \int_{t_1}^{t_2} u_i(x + c_j t, t)dt \tag{2.6}$$

Proposition 2.2.— *Suppose Conditions 1 and 2. Then there exists $p \geqq 0$ such that*

$$I = I_0 \cup I_1 \cup \cdots \cup I_p \tag{2.7}$$

For analyzing closely our partial differential equations system, now we consider simpler equations system ; this is motivated by the dissipation of the effects due to the binary collision terms when time is going to infinity, which is suggested by definability of the wave operator for the system without the linear term, due to Bony [4] :

$$\text{(O)} \qquad \begin{cases} \dfrac{df_i}{dt} = L_i(f) \\ f_i|_{t=0} = f_i^0 > 0 \end{cases}$$

Proposition 2.3.— *Suppose Conditions 1 and 2. 1) for all $i \in I$, $f_i(t)$ is positive.*
2) for $i \in I_0$, $f_i(t)$ is increasing and bounded, so tends to a limit > 0 as $t \to +\infty$.
3) for $i \notin I_0$, $f_i(t)$ tends to 0 exponentially as $t \to +\infty$.

Now we fix t_1 and decompose u_i into the sum of "(quasi-)linear part" v_i and "(essential-)nonlinear part" w_i. Let v_i a solution for the equations system :

$$\text{(V)} \qquad \begin{cases} \left(\dfrac{\partial}{\partial t} + c_i \dfrac{\partial}{\partial x}\right) v_i = L_i(v) - \displaystyle\sum_{j,k,\ell} A_{k\ell}^{ij} u_j \cdot v_i \\ v_i|_{t=t_1} = u_i(\cdot, t_1) \end{cases}$$

where u_i is the solution of (B).

Then $w_i = u_i - v_i$ should satisfy

$$\text{(W)} \qquad \begin{cases} \left(\dfrac{\partial}{\partial t} + c_i \dfrac{\partial}{\partial x}\right) w_i = L_i(w) + Q_i(w) + r_i - s_i \\ w_i|_{t=t_1} = 0 \\ \text{with} \quad r_i = \displaystyle\sum_{j,k,\ell} A_{ij}^{k\ell}(v_k v_\ell + w_k v_\ell + v_k w_\ell) \\ s_i = \displaystyle\sum_{j,k,\ell} A_{k\ell}^{ij} w_i v_j \end{cases}$$

Definition.— *The operator $\mathcal{P} = (\mathcal{P}_i)_i$ is said to be positively preserving if ond only if the solution u_i is nonnegative over $\mathbf{R} \times \mathbf{R}_+$, where $u_i(x,t)$ is a solution for the equations system :*

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + c_i \dfrac{\partial}{\partial x}\right) u_i = \mathcal{P}(u) \\ u_i|_{t=0} = u_i^0(x) \geqq 0 \end{cases} \tag{2.8}$$

Corollary 2.4.— *The operators $(Q_i)_i$ and $(L_i)_i$ are positively preserving.*

Proposition 2.5.—

$$v_i(x,t) \geqq 0 \text{ and } w_i(x,t) \geqq 0 \text{ for any } x \in \mathbf{R} \text{ and } t \in \mathbf{R}_+ \tag{2.9}$$

Now we have some remarks :

Corollary 2.6.—

$$\int_{t_1}^{t_2}\int_{\mathbf{R}} r_i dx dt \leqq \Delta(t_1,t_2), \quad \int_{t_1}^{t_2}\int_{\mathbf{R}} s_i dx dt \leqq \Delta(t_1,t_2) \tag{2.10}$$

We are now going to estimate the "linear part" solution v_i. Its estimation in "L^∞" as follows :

Proposition 2.7.— *The function* $V(t) \equiv \sup_{i,x} \dfrac{v_i(x,t)}{f_i(t)}$ *is strictly decreasing.*

The estimation for v_i and w_i along the characteristic as following :

Proposition 2.8.— *For* $c_i \neq c_j$ *and* $t_1 < t_2$,

$$\sup_{x\in\mathbf{R}} \int_{t_1}^{t_2} v_i(x + c_j t, t) dt \leqq C_* \delta(t_1,t_2) \tag{2.11}$$

$$\sup_{x\in\mathbf{R}} \int_{t_1}^{t_2} w_i(x + c_j t, t) dt \leqq C_* \Delta(t_1,t_2), \tag{2.12}$$

where these constants C_* *depend only on the equations system.*

Now we would like to estimate more closely $M(t_2)$ in terms of $M(t_1)$ for $t_1 < t_2 < T^*$, where

$$M(t) = \max_{i\in I} \sup_{s\leq t} \sup_{x\in\mathbf{R}} u_i(x,s) \quad \text{for} \quad t < T^* \tag{2.13}$$

For the first, we integrate, along a characteristic curve, the equations system for w_i : then

$$\begin{aligned} w_i(t_2, x_*) \leqq C_* \int_{t_1}^{t_2} (\sum_{j\neq i} w_j + \sum_{k,j\neq i} w_k w_j \\ + \sum_{k,j\neq i} v_k v_j + \sum_{k,j\neq i} w_k v_j + \sum_{k,j\neq i} v_k w_j)(x + c_i t, t) dt \end{aligned} \tag{2.14}$$

where $x_* = x + c_i(t_2 - t_1)$. Then we have

$$\int_{t_1}^{t_2} \sum_{j\neq i} w_j(x + c_i t, t) dt \leqq C_* \Delta(t_1,t_2) \tag{2.15}$$

$$\int_{t_1}^{t_2} \sum_{k,j\neq i} w_k w_j(x+c_i t, t)dt \leqq C_* M(t_2)\Delta(t_1,t_2) \tag{2.16}$$

$$\int_{t_1}^{t_2} \sum_{k,j\neq i} v_k v_j(x+c_i t, t)dt \leqq C_* M(t_2)\delta(t_1,t_2) \tag{2.17}$$

Hence we have

$$\sup_x w_i(x,t_2) \leqq C_*\left(1+M(t_2)\right)\cdot\Delta(t_1,t_2) + C_* M(t_2)\delta(t_1,t_2) \tag{2.18}$$

Consequently for $u_i(x,t)$, we have

$$\begin{aligned} \sup_x u_i(x,t_2) &\leqq \sup_x v_i(x,t_2) + \sup_x w_i(x,t_2) \\ &\leqq C_* M(t_1) + C_*\left(1+M(t_2)\right)\cdot\Delta(t_1,t_2) + C_* M(t_2)\delta(t_1,t_2) \end{aligned} \tag{2.19}$$

We take $T < T^*$ and thereafter a sequence $0 = t_0 < t_1 < \cdots < t_N = T$ such that $\Delta(t_j,t_{j+1}) < (4C_*)^{-1}$ and $\delta(t_j,t_{j+1}) < (4C_*)^{-1}$. Then, seeing that $\Delta(0,T) \leqq C_*\mu^2$ and $\delta(0,T) \leqq C_*\mu$ by virtue of Lemma 2.1, we have $N = O(\mu^2+\mu)$, and

$$\begin{aligned} M(t_{j+1}) &\leqq C_* M(t_j) + C_*\Delta(t_j,t_{j+1}) \\ &\leqq C_* M(t_j) + C_* \end{aligned} \tag{2.20}$$

therefore we obtain

$$u_i(x,t) \leqq (1+\sup_{i,x} u_i^0)\exp(a\mu^2+b\mu) \tag{2.21}$$

§2.2 Proof of the theorem 3

We examine in a more detailed way the argument developed in the last chapter, that is, to decompose the solution u_i into the sum of "(quasi-)linear part" v_i and "(essential-) nonlinear part w_i. Later on, we specify t_1, which will be noted T, and the dependence of v_i and w_i on a cut time T. Let's write down u_i of the form $u_i = v_i^T + w_i^T$:

$$(V^T) \qquad \begin{cases} \left(\dfrac{\partial}{\partial t} + c_i \dfrac{\partial}{\partial x}\right) v_i^T = L_i(v^T) - \displaystyle\sum_{j,k,\ell} A_{k\ell}^{ij} u_j \cdot v_j^T \\ v_i^T|_{t=T} = u_i(\cdot, T) \end{cases}$$

$$(W^T) \qquad \begin{cases} \left(\dfrac{\partial}{\partial t} + c_i \dfrac{\partial}{\partial x}\right) w_i^T = L_i(w^T) + Q_i(w^T) + r_i^T - s_i^T \\ w_i^T|_{t=T} = 0 \\ \text{with} \quad r_i^T = \displaystyle\sum_{j,k,\ell} A_{ij}^{k\ell}(v_k^T v_\ell^T + w_k^T v_\ell^T + v_k^T w_\ell^T) \\ \qquad\quad s_i^T = \displaystyle\sum_{j,k,\ell} A_{k\ell}^{ij} w_i^T v_j^T \end{cases}$$

Knowing that

$$u_i(x,t) \leqq M \equiv (1 + \sup_{i,x} u_i^0) \exp(a\mu^2 + b\mu), \tag{2.22}$$

we have, for $t > T$

$$\begin{aligned} \sup_x w_i^T(x,t) &\leqq C_* (1 + M(t)) \cdot \Delta(T,t) + C_* M(t) \delta(T,t) \\ &\leqq C_*(1+M)(\Delta(T,t) + \delta(T,t)) \end{aligned} \tag{2.23}$$

$$\left\| \sum_i w_i^T(\cdot,t) \right\|_{L^1} \leqq C_* \Delta(T,t) \tag{2.24}$$

Using Lemma 2.1. which says $\Delta(0,\infty) \leqq C_* \mu^2$ and $\delta(0,\infty) \leqq C_* \mu$, we conclude that, for any $\varepsilon > 0$, there exists T such that

$$\Delta(T,\infty) + \delta(T,\infty) < C_* [(1+M) \times (\sharp\{i \in I\})]^{-1} \cdot \varepsilon \tag{2.25}$$

Then we have, for $t > T$,

$$\left\| \sum_i w_i^T(\cdot,t) \right\|_{L^1 \cap L^\infty} < \varepsilon, \tag{2.26}$$

hence we have

$$\sup_i \left\| w_i^T(\cdot,t) \right\|_{L^p} < \varepsilon \ (1 \leqq p \leqq \infty) \tag{2.27}$$

i.e.

$$\sup_i \left\| u_i(x,t) - v_i^T(x,t) \right\|_{L^p} < \varepsilon \ (1 \leqq p \leqq \infty), \tag{2.28}$$

save for trivial constants. Consequently we prove the first assertion of the theorem.

Now we would like to estimate v_i :

Proposition 2.9.— *1)* $V(t) = \sup_{i,x} \dfrac{v_i^T(x,t)}{f_i(t)}$ *is strictly decreasing.*

2) $\sum_i \|V_i(\cdot,t)\|_{L^2}^2$ *is also strictly decreasing where* $V_i(x,t) \equiv \dfrac{v_i^T(x,t)}{\sqrt{f_i(t)}}$.

We pursue the proof of the theorem. By the above proposition, we have

$$\max_i \left\| \frac{v_i^T(\cdot,t)}{f_i(t)} \right\|_{L^\infty} \quad \text{and} \quad \max_i \left\| \frac{v_i^T(\cdot,t)}{f_i(t)} \right\|_{L^2} \tag{2.29}$$

are strictly decreasing, where a positive bounded functions $f_i(t)$ verifies the following condition :
- for $i \in I_0$, $f_i(t)$ is increasing and tends to a limit > 0 as $t \to +\infty$.
- for $i \notin I_0$, $f_i(t)$ tends to 0 exponentially as $t \to +\infty$. The interpolation between L^2 and L^∞ achieves then the proof.

§2.3 Proof of the theorem 5

Let's consider the following equations system with ε :

$$(B_\varepsilon) \qquad \begin{cases} \dfrac{\partial u_i}{\partial t} + c_i \dfrac{\partial u_i}{\partial x} = \varepsilon Q_i(u) + L_i(u) \\ u_i|_{t=0} = u_i^0(\cdot) \end{cases}$$

where ε is a positive constant.

For this Cauchy problem, we would like to seek a solution $u_i(x,t)$ of type $u_i = \sum_{m=0}^{\infty} \varepsilon^m u_i^{(m)}$ (so-called Hilbert-Chapman-Enskog expansion).

Remark : From a physical point of view, ε corresponds to the inverse of the "Knudsen Number" and $\varepsilon \to 0$ corresponds to a fluid dynamical limit to a free molecular flow.

To prove Theorem 5, it is sufficient to show the following theorem :

Theorem 2.10.— *Suppose Conditions 5 and 6. For* $\varepsilon \in \left[0, C_{**} \left(\sum_{k=0}^{s} J_k^2\right)^{-\frac{1}{2}}\right[$, *the series* $u_i = \sum_{m=0}^{\infty} \varepsilon^m u_i^{(m)}$ *converge in* $H^s (s = 1, 2, \cdots)$, *so* L^∞, *uniformly with respect to* $t \in \mathbf{R}_+$ *and then*

$$\|u_i\|_{H^s} (\text{ so } \|u_i\|_{L^\infty}) \leqq \begin{cases} C_* \left(\sum_{k=0}^{s} J_k^2\right)^{\frac{1}{2}} & i \in I_0 \\ C_* \left(\sum_{k=0}^{s} J_k^2\right)^{\frac{1}{2}} e^{-\frac{1}{2}\lambda t} & i \in I_1 \end{cases} \tag{2.30}$$

where $J_s = \left(\sum_i \|D^s u_i^0\|_{L^2}^2\right)^{\frac{1}{2}}$, *constants* C_* *and* C_{**} *depend only on the equations system and constant* $\lambda > 0$.

Like as in Chapter 6, we use

$$\text{(O)}\qquad \begin{cases} \dfrac{df_i}{dt} = L_i(f) \\ f_i|_{t=0} = f_i^0 > 0 \end{cases}$$

Condition 6 implies a more precise estimation for f_i than Proposition 2.3. :

Proposition 2.11.— *Suppose Conditions 5 and 6. There exists $f_i^0 > 0$ such that, for $i \in I_1$, f_i tends to 0 with the same order i.e. there is $\lambda > 0$ such that $f_i(t) = e^{-\lambda t}P_i(t)$ with P_i polynomial of t and for $i \in I_0$, $f_i(t)$ is increasing and bounded, so tends to a limit > 0 as $t \longrightarrow +\infty$.*

Let put $w_i(x,t) = \dfrac{u_i(x,t)}{\sqrt{f_i(t)}}$. Now we write down the equation for $w_i(x,t)$, and put $w_i(x,t) = \sum_{i=0}^{\infty} \varepsilon^m w_i^{(m)}(x,t)$. Then we have for $m = 0,1,2,\cdots$

$$\text{(2.31)}\qquad \begin{cases} \left(\dfrac{\partial}{\partial t} + c_i \dfrac{\partial}{\partial x}\right) w_i^{(m)} = \displaystyle\sum_k \left\{ \alpha_i^k \left(\frac{f_k}{f_i}\right)^{\frac{1}{2}} w_k^{(m)} - \alpha_k^i w_i^{(m)} \right\} \\ \qquad\qquad - \dfrac{L_i(f)}{2f_i} w_i^{(m)} + F_i^{(m)}(w) \\ w_i^{(m)}|_{t=0} = \begin{cases} u_i^0(x), & m = 0 \\ 0, & m = 1,2,\cdots \end{cases} \end{cases}$$

where

$$\text{(2.32)}\qquad F_i^{(m)}(w) = \begin{cases} \displaystyle\sum_{n=0}^{m-1} \sum_{jk} B_i^{jk} \left(\frac{f_j f_k}{f_i}\right)^{\frac{1}{2}} w_j^{(n)} w_k^{(m-n-1)} & \text{for } m = 1,2,\cdots \\ 0 & \text{for } m = 0 \end{cases}$$

The energy estimation leads us :

Proposition 2.12.— *Suppose Conditions 5 and 6. For $s = 0,1,2,\cdots$*

$$\text{(2.33)}\qquad \begin{aligned} \frac{d}{dt} \sum_i \left\| D^s w_i^{(m)} \right\|_{L^2}^2 &= - \sum_{ik} \alpha_i^k f^k \left\| D^s \left(\frac{w_i^{(m)}}{f_i^{\frac{1}{2}}} - \frac{w_k^{(m)}}{f_k^{\frac{1}{2}}} \right) \right\|_{L^2}^2 \\ &\quad + 2 \sum_i \left\{ \left(D^s F_i^{(m)}(w), D^s w_i^{(m)} \right)_{L^2} \right\} \end{aligned}$$

especially for $m = 0$,

$$\text{(2.34)}\qquad \frac{d}{dt} \sum_i \left\| D^s w_i^{(0)} \right\|_{L^2}^2 = - \sum_{ik} \alpha_i^k f^k \left\| D^s \left(\frac{w_i^{(0)}}{f_i^{\frac{1}{2}}} - \frac{w_k^{(0)}}{f_k^{\frac{1}{2}}} \right) \right\|_{L^2}^2 \leqq 0$$

Corollary 2.13.— *Suppose Conditions 5 and 6. Then we have, for $s = 0,1,2,\cdots$*

$$\text{(2.35)}\qquad \left\| D^s w_i^{(0)} \right\|_{L^2} \leqq C_* J_s \text{ for all } i$$

where $J_s = \left(\sum_i \left\| D^s u_i^0 \right\|_{L^2}^2 \right)^{\frac{1}{2}}$.

Proposition 2.14.— *Suppose Conditions 5 and 6. For* $s = 1, 2, \cdots$

$$\int_0^\infty \sum_i \left\| F_i^{(1)}(w) \right\|_{H^s} dt \leqq C_* \sum_{k=0}^{s} J_k^2 \tag{2.36}$$

Proposition 2.15.— *Suppose Conditions 5 and 6. Then we have, for* $s = 1, 2, \cdots$

$$\left\| w_i^{(1)} \right\|_{H^s} \leqq C_* \sum_{k=0}^{s} J_k^2 \quad \text{for all} \quad i \tag{2.37}$$

Now let's put $\left\| w_i^{(m)} \right\|_{H^s} \leqq a_s^{(m)}$ for $s = 1, 2, \cdots$, then we have by induction

$$a_s^{(m+1)} \leqq C_* b_s^{(m)} \equiv C_* \sum_{n=0}^{m} a_s^{(n)} a_s^{(m-n)} \tag{2.38}$$

Let's put $f(x) = f_s(x) = \sum_{n=0}^{\infty} a_s^{(n)} x^n$, the above inequality means that

$$\frac{f(x) - f(0)}{x} = C_* \{f(x)\}^2, \tag{2.39}$$

then we have

$$f(x) = \frac{1 - \sqrt{1 - 4C_* x f(0)}}{2C_* x}, \quad f(0) = C_* \left(\sum_{k=0}^{s} J_k^2 \right)^{\frac{1}{2}} \tag{2.40}$$

It is easy to see that the right-hand side can be written in infinite series with a positive convergence radius, which achieve the proof.

§2.4 Proof of Theorem 6

Like as in the previous section, we consider the following equations system with ε :

$$(M_\varepsilon) \qquad \begin{cases} \dfrac{\partial u_i}{\partial t} + c_i \dfrac{\partial u_i}{\partial x} = \varepsilon R_i(u) + L_i(u) \\ u_i|_{t=0} = u_i^0(\cdot) \end{cases}$$

where ε is a positive constant.

The same argument shows

$$\left\| w_i^{(m)} \right\|_{H^s} \leqq b_s^{(m)} \text{ for } s = 1, 2, \cdots \tag{2.41}$$

where

$$\begin{cases} b_s^{(m+1)} = C \displaystyle\sum_{p=2}^{\sigma} \sum_{n_1 + \cdots n_p = m} b_s^{(n_1)} \cdots b_s^{(n_p)} \\ b_s^{(0)} = C \left(\displaystyle\sum_{k=0}^{s} J_k^2 \right)^{\frac{1}{2}} \end{cases} \tag{2.42}$$

Similarly we achieve the proof.

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