数理解析研究所講究録
第 834巻 1993 年 70-79


# THE YAMABE PROBLEM AND NONLINEAR BOUNDARY VALUE PROBLEMS

KAZUAKI TAIRA (平良 和昭)

Institute of Mathematics, University of Tsukuba, Tsukuba 305, Japan

ABSTRACT. We study the Yamabe problem in the context of manifolds with boundary - a basic problem in Riemannian geometry - from the point of view of nonlinear elliptic boundary value problems. By making good use of bifurcation theory from a simple eigenvalue, we show that nonpositive scalar curvatures and nonpositive mean curvatures are not always conformal to constant negative scalar curvatures and the zero mean curvature.

## 1. INTRODUCTION

Let $(\overline{M}, g)$ be a smooth compact, connected Riemannian manifold with boundary $\partial M$ of dimension $n \geq 3$, and let $M = \overline{M} \backslash \partial M$ be the interior of $\overline{M}$. A basic problem in Riemannian geometry is to seek a conformal change of the metric $g$ that makes the scalar curvature of $M$ *constant* and the mean curvature of $\partial M$ *zero*. When the boundary $\partial M$ is empty, this problem is the so-called Yamabe problem. The solution of the Yamabe problem is completely given by H. Yamabe [Y], N. S. Trudinger [Tr], T. Aubin [Au] and R. Schoen [S] (cf. [LP]). Recently, J. Escobar [E] has studied the problem in the context of manifolds with boundary, and has given an affirmative solution to the problem formulated above in almost every case.

In this paper we consider the case where the given metric $g$ *already* has a constant negative scalar curvature $k$ of $M$ and the zero mean curvature of $\partial M$ as in Ouyang [O] (cf. [K], [KW]). Our problem is the following:

**Problem.** *Given a nonpositive smooth function $R'$ in $M$ and a nonpositive smooth function $h'$ on $\partial M$, find a metric $g'$ of $\overline{M}$, conformal to $g$, such that $R'$ and $h'$ are the scalar curvature of $M$ and the mean curvature of $\partial M$ with respect to $g'$, respectively.*

We shall show that nonpositive scalar curvatures $R'$ and nonpositive mean curvatures $h'$ are not always conformal to negative scalar curvatures $k$ and the zero mean curvature; it depends on the *shape* of the zero set of $R'$ (see Main Theorem below).

If $g_{jk}$ are the components of the metric tensor $g$ with respect to a local coordinate system $x^1, \cdots, x^n$, then $g_{jk}$ and its inverse $g^{jk}$ are used to raise and lower indices.

---

1991 *Mathematics Subject Classification.* Primary 53A30; Secondary 35J60, 35B32.
*Key words and phrases.* Yamabe problem, nonlinear boundary value problems, bifurcation.

Covariant differentiation is denoted by $\nabla$. If $f$ is a function on $M$, then its covariant derivative is the one-tensor $\nabla f$ with components

$$\nabla_i f = \frac{\partial f}{\partial x^i}.$$

The second covariant derivative of $f$ is the two-tensor $\nabla^2 f$ with components

$$\nabla_{ij} f = \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{\ell=1}^{n} \Gamma_{ij}^{\ell} \frac{\partial f}{\partial x^\ell}.$$

Here the functions

$$\Gamma_{ij}^{\ell} = \frac{1}{2}\left[\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right] g^{k\ell}$$

are the *Christoffel symbols*. The metric extends to an inner product on tensors of any type; for example, the norm of $\nabla f$ is

$$|\nabla f|^2 = \sum_{j=1}^{n} \nabla^j f \nabla_j f = \sum_{i,j=1}^{n} g^{ij} \nabla_i f \nabla_j f.$$

The *divergence operator* is the formal adjoint $\nabla^*$ of $\nabla$ given on one-forms $u = \sum_{i=1}^{n} u_i\, dx^i$ by

$$\nabla^* u = -\sum_{i=1}^{n} \nabla^i u_i = -\sum_{i,j=1}^{n} g^{ij} \nabla_j u_i = -\sum_{i,j=1}^{n} g^{ij} \frac{\partial u_i}{\partial x^j} + \sum_{i,j,\ell=1}^{n} g^{ij}\, \Gamma_{ji}^{\ell}\, u_\ell.$$

The *Laplace-Beltrami operator*, or simply *Laplacian*, is the second-order differential operator $\Delta$ given on functions $f$ by

$$\Delta f = \nabla^* \nabla f = -\sum_{i=1}^{n} \nabla^i \nabla_i f = -\sum_{i,j=1}^{n} g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + \sum_{i,j,\ell=1}^{n} g^{ij}\, \Gamma_{ji}^{\ell} \frac{\partial f}{\partial x^\ell}.$$

The Riemannian *curvature tensor* is the tensor with components $R^{\ell}{}_{kij}$ computed in a local coordinate system $x^1, \cdots, x^n$ by

$$R^{\ell}{}_{kij} = \frac{\partial}{\partial x^i}(\Gamma^{\ell}{}_{jk}) - \frac{\partial}{\partial x^j}(\Gamma^{\ell}{}_{ik}) + \sum_{m=1}^{n} \Gamma^{\ell}{}_{im}\, \Gamma^{m}{}_{jk} - \sum_{m=1}^{n} \Gamma^{\ell}{}_{jm}\, \Gamma^{m}{}_{ik}.$$

The *Ricci tensor* is the contraction of the curvature tensor

$$R_{ij} = \sum_{k=1}^{n} R^{k}{}_{ikj},$$

and the *scalar curvature* is the trace of the Ricci tensor

$$R = \sum_{i,j=1}^{n} g^{ij} R_{ij}.$$

Let $(x^1, \cdots, x^{n-1}, x^n)$ be a local coordinate system on $\overline{M}$ in which $\partial M$ is the plane $x^n = 0$ and for which $\partial/\partial x^n$ is a unit outward normal vector to $\partial M$. Then the components $h_{ij}$ of the second fundamental form of $g$ are given by

$$h_{ij} = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^n}, \quad 1 \leq i, j \leq n-1.$$

The *mean curvature* of $\partial M$ is the trace

$$h = \frac{1}{n-1}\sum_{i,j=1}^{n-1} g^{ij} h_{ij}.$$

A metric $g'$ of $\overline{M}$ is said to be *conformal* to the metric $g$ if there exists a smooth real-valued function $f$ on $\overline{M}$ such that

$$g' = e^{2f} g.$$

If $g' = e^{2f} g$ is a metric conformal to $g$, then we have the following transformation laws for the Ricci curvatures $R_{ij}$, $R'_{ij}$ and the scalar curvatures $R$, $R'$, respectively:

$$R'_{ij} = R_{ij} - (n-2)\nabla_{ij} f + (n-2)\nabla_i f \nabla_j f + \left(\Delta f - (n-2)|\nabla f|^2\right) g_{ij},$$
$$R' = e^{-2f}\left(R + 2(n-1)\Delta f - (n-1)(n-2)|\nabla f|^2\right).$$

Furthermore, if we make the substitution $e^{2f} = \varphi^{4/(n-2)}$, $\varphi > 0$ on $\overline{M}$, then the second formula can be simplified as follows:

$$4\frac{n-1}{n-2}\Delta\varphi + R\varphi - R'\varphi^{\frac{n+2}{n-2}} = 0. \tag{1}$$

Similarly, one can compute the components $h'_{ij}$ of the second fundamental form of $g' = e^{2f} g$ in terms of the second fundamental form of $g$. We have the following transformation laws for the components $h_{ij}$, $h'_{ij}$ and the mean curvatures $h$, $h'$, respectively:

$$h'_{ij} = e^f h_{ij} + \frac{\partial}{\partial \mathbf{n}}(e^f) g_{ij}, \qquad h' = e^{-f}\left(h + \frac{\partial f}{\partial \mathbf{n}}\right),$$

where $\partial/\partial \mathbf{n}$ is the unit outward normal derivative. Furthermore, if we make the substitution $e^{2f} = \varphi^{4/(n-2)}$ as above, then the second formula can be simplified as follows:

$$\frac{2}{n-2}\frac{\partial \varphi}{\partial \mathbf{n}} + h\varphi - h'\varphi^{\frac{n}{n-2}} = 0. \tag{2}$$

Therefore, if we take $R = k$ in equation (1) and $h = 0$ in condition (2), our problem is equivalent to finding a smooth strictly positive solution $\varphi$ on $\overline{M}$ of the nonlinear boundary value problem:

$$(*) \qquad \begin{cases} 4\frac{n-1}{n-2}\Delta\varphi + k\varphi - R'\varphi^{\frac{n+2}{n-2}} = 0 & \text{in } M, \\ \frac{2}{n-2}\frac{\partial \varphi}{\partial \mathbf{n}} - h'\varphi^{\frac{n}{n-2}} = 0 & \text{on } \partial M. \end{cases}$$

Now we assume that

$$R' \leq 0 \quad \text{in } M.$$

We let

$$\mathcal{M}_-(R') = \{x \in M; R'(x) < 0\},$$

and

$$\mathcal{M}_0(R') = M \backslash \overline{\mathcal{M}_-(R')}.$$

Our fundamental hypothesis is the following (cf. Figure 1):

$(H)$ The open set $\mathcal{M}_0(R')$ consists of a finite number of connected components with smooth boundary, say $\mathcal{M}_i(R')$, $1 \leq i \leq \ell$, which are bounded away from $\partial M$, and of a finite number of connected components with smooth boundary, say $\mathcal{M}_j(R')$, $\ell+1 \leq j \leq N$, such that each closure $\overline{\mathcal{M}_j(R')}$ is a neighborhood of some connected component $\mathcal{S}_j$ of $\partial M$.

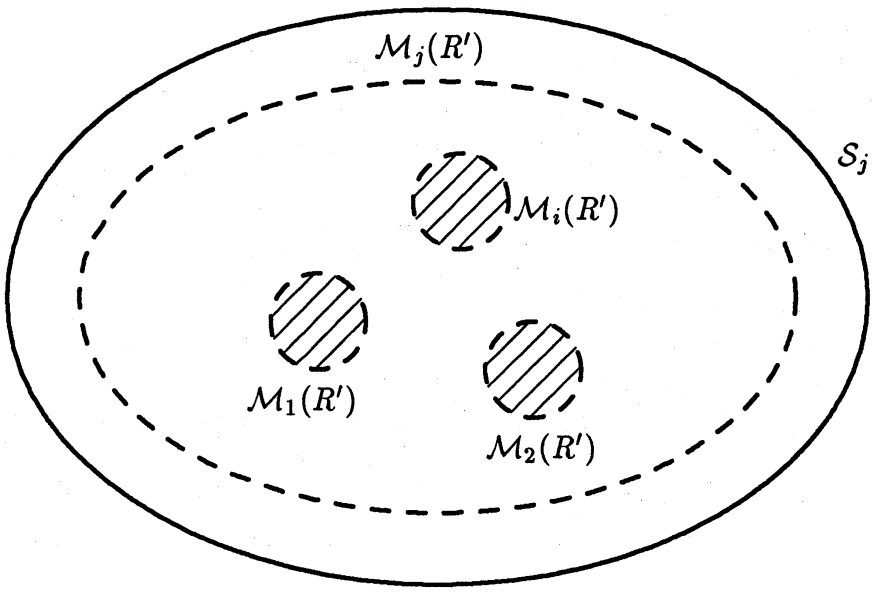


Figure 1

First we consider the Dirichlet eigenvalue problem in each connected component $\mathcal{M}_i(R')$, $1 \leq i \leq \ell$, which is bounded away from $\partial M$:

$$(D_i) \qquad \begin{cases} \Delta\psi = \lambda\psi & \text{in } \mathcal{M}_i(R'), \\ \psi = 0 & \text{on } \partial\mathcal{M}_i(R'). \end{cases}$$

By the celebrated Rayleigh theorem (cf. [Ag, Chapter 10], [Cl, Chapter I]), we know that the *first eigenvalue* $\lambda_1(\mathcal{M}_i(R'))$ of problem $(D_i)$ is given by the formula

$$\lambda_1(\mathcal{M}_i(R')) = \inf\left\{\int_{\mathcal{M}_i(R')} |\nabla\psi|^2 \, dV; \psi \in H_0^1(\mathcal{M}_i(R')), \|\psi\|_{L^2(\mathcal{M}_i(R'))} = 1\right\}.$$

Here $dV$ is the Riemannian density of $g$, and $H_0^1(\mathcal{M}_i(R'))$ is the closure of smooth functions with compact support in $\mathcal{M}_i(R')$ in the Sobolev space $H^1(\mathcal{M}_i(R'))$.

Next we consider the Dirichlet-Neumann eigenvalue problem in each connected component $\mathcal{M}_j(R')$, $\ell+1 \le j \le N$, whose closure is a neighborhood of some connected component $\mathcal{S}_j$ of $\partial M$:

$$(M_j) \qquad \begin{cases} \Delta\psi = \mu\psi & \text{in } \mathcal{M}_j(R'), \\ \psi = 0 & \text{on } \partial\mathcal{M}_j(R')\backslash\mathcal{S}_j, \\ \frac{\partial\psi}{\partial \mathbf{n}} = 0 & \text{on } \mathcal{S}_j. \end{cases}$$

Similarly, by Rayleigh's theorem, we know that the *first eigenvalue* $\mu_1(\mathcal{M}_j(R'))$ of problem $(M_j)$ is given by the formula

$$\mu_1(\mathcal{M}_j(R')) = \inf\Big\{\int_{\mathcal{M}_j(R')} |\nabla\psi|^2\, dV;\ \psi \in H^1(\mathcal{M}_j(R')), \psi = 0 \text{ on } \partial\mathcal{M}_j(R')\backslash\mathcal{S}_j,$$
$$\|\psi\|_{L^2(\mathcal{M}_j(R'))} = 1\Big\}.$$

We let

$$\widetilde{\lambda}_1(\mathcal{M}_0(R')) = \min\{\lambda_1(\mathcal{M}_1(R')), \cdots, \lambda_1(\mathcal{M}_\ell(R')),$$
$$\mu_1(\mathcal{M}_{\ell+1}(R')), \cdots, \mu_1(\mathcal{M}_N(R'))\}.$$

Then our main result of this paper is stated as follows.

**Main Theorem.** *Assume that the given metric $g$ has a constant negative scalar curvature $k$ of $M$ and the zero mean curvature of $\partial M$, and that:*

*(A) $R' \le 0$ in $M$.*

*(H) The open set $\mathcal{M}_0(R')$ consists of a finite number of connected components $\mathcal{M}_i(R')$, $1 \le i \le \ell$, with smooth boundary which are bounded away from $\partial M$, and of a finite number of connected components $\mathcal{M}_j(R')$, $\ell+1 \le j \le N$, with smooth boundary such that each closure $\overline{\mathcal{M}_j(R')}$ is a neighborhood of some connected component $\mathcal{S}_j$ of $\partial M$.*

*(B) $h' \le 0$ on $\partial M\backslash\mathcal{S}_j$, and $h' = 0$ on $\mathcal{S}_j$, $\ell+1 \le j \le N$.*

*Then we have the following:*

*(i) If the zero set $\mathcal{M}_0(R')$ is so small that*

$$\widetilde{\lambda}_1(\mathcal{M}_0(R')) > -\frac{n-2}{4(n-1)}k,$$

*then there exists a conformally related metric $g' = \varphi^{4/(n-2)}g$, $\varphi > 0$ on $\overline{M}$, such that $R'$ and $h'$ are the scalar curvature of $M$ and the mean curvature of $\partial M$ with respect to $g'$, respectively.*

*(ii) If the zero set $\mathcal{M}_0(R')$ is so large that*

$$\widetilde{\lambda}_1(\mathcal{M}_0(R')) \le -\frac{n-2}{4(n-1)}k,$$

*then there exists no such conformal metric $g'$.*

## 2. Outline of Proof

If we let

$$\lambda = -\frac{n-2}{4(n-1)}k, \quad h = -\frac{n-2}{4(n-1)}R', \quad a = -\frac{n-2}{2}h',$$

then our problem (*) can be written in the following form:

$$(**) \qquad \begin{cases} \Delta u - \lambda u + hu^p = 0 & \text{in } M, \\ \frac{\partial u}{\partial \mathbf{n}} + au^q = 0 & \text{on } \partial M, \end{cases}$$

where

$$p = \frac{n+2}{n-2} > 1, \quad q = \frac{n}{n-2} > 1.$$

We remark that

$$\begin{cases} \lambda > 0, \\ h \geq 0 & \text{in } M, \\ a \geq 0 & \text{on } \partial M. \end{cases}$$

Now we free our problem from geometry, and study the existence and nonexistence of positive solutions of problem (**) in the framework of Hölder spaces. Our approach to problem (**) is a modification of that of Ouyang [O] adapted to the present context. However we do not use the sub-super-solution method as in Ouyang [O] (cf. [K], [KW]).

Our proof of Main Theorem is based on the following bifurcation theorem from a simple eigenvalue due to Crandall-Rabinowitz [CR]:

**The bifurcation theorem.** *Let $X$, $Y$ be Banach spaces, and let $V$ be a neighborhood of 0 in $X$ and let $F : (-1,1) \times V \to Y$ have the following properties:*

*(1) $F(t,0) = 0$ for $|t| < 1$.*

*(2) The partial Fréchet derivatives $F_t$, $F_x$ and $F_{tx}$ of $F$ exist and are continuous.*

*(3) $N(F_x(0,0))$ and $Y/R(F_x(0,0))$ are one dimensional.*

*(4) $F_{tx}(0,0)x_0 \notin R(F_x(0,0))$ where $N(F_x(0,0)) = \text{span}\{x_0\}$.*

*If $Z$ is a complement of $N(F_x(0,0))$ in $X$, that is, if it is a closed subspace of $X$ such that*

$$X = N(F_x(0,0)) \oplus Z,$$

*then there exist a neighborhood $U$ of $(0,0)$ in $\mathbf{R} \times X$ and an open interval $(-a,a)$ such that the set of solutions of $F(t,x) = 0$ in $U$ consists of two continuous curves $\Gamma_1$ and $\Gamma_2$ which may be parametrized by $t$ and $\alpha$ as follows (cf. Figure 2):*

$$\Gamma_1 = \{(t,0); (t,0) \in U\},$$
$$\Gamma_2 = \{(\varphi(\alpha), \alpha x_0 + \alpha\psi(\alpha)); |\alpha| < a\}.$$

*Here*

$$\varphi : (-a,a) \to \mathbf{R}, \quad \varphi(0) = 0,$$
$$\psi : (-a,a) \to Z, \quad \psi(0) = 0.$$

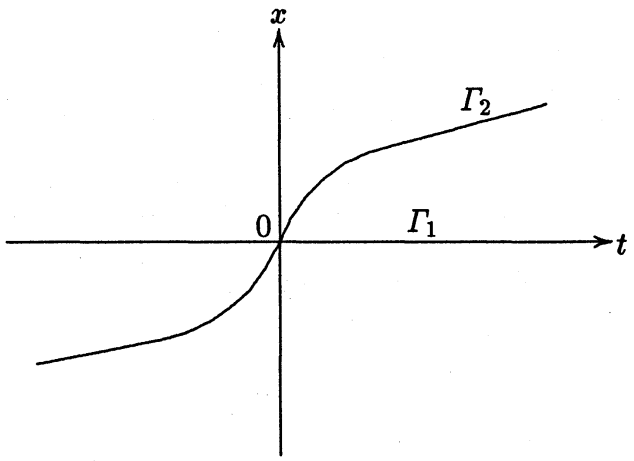


Figure 2

1) First we associate with problem (**) a nonlinear mapping $F : \mathbf{R} \times C^{2+\theta}(\overline{M}) \mapsto C^{\theta}(\overline{M}) \times C^{1+\theta}(\partial M)$ $(0 < \theta < 1)$ as follows:

$$F : \mathbf{R} \times C^{2+\theta}(\overline{M}) \longrightarrow C^{\theta}(\overline{M}) \times C^{1+\theta}(\partial M)$$
$$(\lambda, u) \longmapsto \left(\Delta u - \lambda u + hu^p, \frac{\partial u}{\partial \mathbf{n}} + au^q\right).$$

We remark that a function $u \in C^{2+\theta}(\overline{M})$ is a solution of problem (**) if and only if $F(\lambda, u) = 0$.

Then we have for partial Fréchet derivatives of $F$

$$F_u(\lambda, u) : C^{2+\theta}(\overline{M}) \longrightarrow C^{\theta}(\overline{M}) \times C^{1+\theta}(\partial M)$$
$$v \longmapsto \left(\Delta v - \lambda v + phu^{p-1}v, \frac{\partial v}{\partial \mathbf{n}} + qau^{q-1}v\right),$$

and

$$F_{\lambda u}(\lambda, u) : C^{2+\theta}(\overline{M}) \longrightarrow C^{\theta}(\overline{M}) \times C^{1+\theta}(\partial M)$$
$$v \longmapsto (-v, 0).$$

In particular we have

$$F_u(0,0) : C^{2+\theta}(\overline{M}) \longrightarrow C^{\theta}(\overline{M}) \times C^{1+\theta}(\partial M)$$
$$v \longmapsto \left(\Delta v, \frac{\partial v}{\partial \mathbf{n}}\right).$$

It is easy to see that

$$N(F_u(0,0)) = \{\text{constant functions}\} = \text{span}\{1\},$$
$$R(F_u(0,0)) = \{(f, \varphi) \in C^{\theta}(\overline{M}) \times C^{1+\theta}(\partial M); \int_M f\, dV + \int_{\partial M} \varphi\, d\sigma = 0\},$$

and

$$F_{\lambda u}(0,0)1 = (-1,0) \notin R(F_u(0,0)).$$

Therefore, by using the bifurcation theorem, we obtain that there exists a bifurcation solution curve $(\lambda, u(\lambda))$ of the equation $F(\lambda, u) = 0$ starting at $(0,0)$.

2) Next, by virtue of the implicit function theorem, we can find a constant $0 < \overline{\lambda}(h) \leq \infty$ such that the Fréchet derivative

$$F_u(\lambda, u(\lambda)) : C^{2+\theta}(\overline{M}) \longrightarrow C^{\theta}(\overline{M}) \times C^{1+\theta}(\partial M)$$

is an algebraic and topological isomorphism for all $0 < \lambda < \overline{\lambda}(h)$. This means that there occurs no *secondary bifurcation* along the bifurcation solution curve $(\lambda, u(\lambda))$ of problem $(**)$ for all $0 < \lambda < \overline{\lambda}(h)$. In the proof we make essential use of the positivity of the resolvent associated with $F_u(\lambda, u(\lambda))$ on the space $C(\overline{M})$ due to Taira [Ta]. Furthermore we show that the solution $u(\lambda)$ "blows up" at the critical value $\overline{\lambda}(h)$. Our situation may be represented schematically by the following bifurcation diagram:

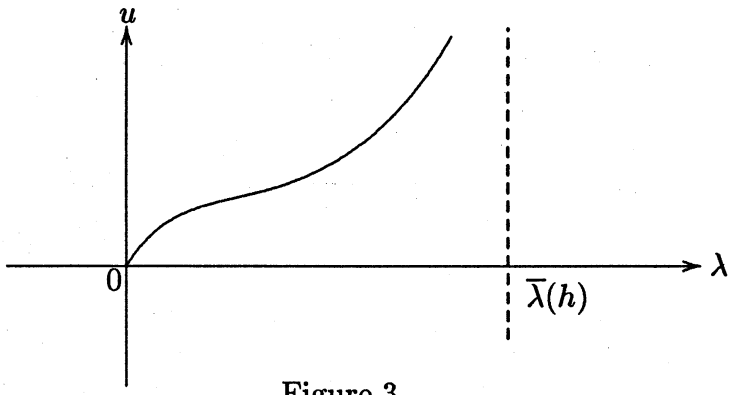


Figure 3

3) In order to characterize the critical value $\overline{\lambda}(h)$ of $\lambda$, we let

$$\mathcal{M}_+(h) = \{x \in M; h(x) > 0\},$$

and

$$\mathcal{M}_0(h) = M \backslash \overline{\mathcal{M}_+(h)}.$$

Our fundamental hypothesis is the following (cf. hypothesis $(H)$):

$(\eta)$ The open set $\mathcal{M}_0(h)$ consists of a finite number of connected components with smooth boundary, say $\mathcal{M}_i(h)$, $1 \leq i \leq \ell$, which are bounded away from $\partial M$, and of a finite number of connected components with smooth boundary, say $\mathcal{M}_j(h)$, $\ell + 1 \leq j \leq N$, such that each closure $\overline{\mathcal{M}_j(h)}$ is a neighborhood of some connected component $\mathcal{S}_j$ of $\partial M$.

We consider the Dirichlet eigenvalue problem in each connected component $\mathcal{M}_i(h)$, $1 \leq i \leq \ell$, which is bounded away from $\partial M$:

$$(D_i) \qquad \begin{cases} \Delta\varphi = \lambda\varphi & \text{in } \mathcal{M}_i(h), \\ \varphi = 0 & \text{on } \partial\mathcal{M}_i(h). \end{cases}$$

The first eigenvalue $\lambda_1(\mathcal{M}_i(h))$ of problem $(D_i)$ is given by the formula

$$\lambda_1(\mathcal{M}_i(h)) = \inf\left\{\int_{\mathcal{M}_i(h)} |\nabla\varphi|^2\, dV; \varphi \in H_0^1(\mathcal{M}_i(h)), \|\varphi\|_{L^2(\mathcal{M}_i(h))} = 1\right\}.$$

We consider the Dirichlet-Neumann eigenvalue problem in each connected component $\mathcal{M}_j(h)$, $\ell+1 \le j \le N$, whose closure is a neighborhood of some connected component $\mathcal{S}_j$ of $\partial M$:

$$(M_j) \qquad \begin{cases} \Delta\varphi = \mu\varphi & \text{in } \mathcal{M}_j(h), \\ \varphi = 0 & \text{on } \partial\mathcal{M}_j(h)\backslash\mathcal{S}_j, \\ \frac{\partial\varphi}{\partial \mathbf{n}} = 0 & \text{on } \mathcal{S}_j. \end{cases}$$

The first eigenvalue $\mu_1(\mathcal{M}_j(h))$ of problem $(M_j)$ is given by the formula

$$\mu_1(\mathcal{M}_j(h)) = \inf\left\{\int_{\mathcal{M}_j(h)} |\nabla\varphi|^2\, dV; \varphi \in H^1(\mathcal{M}_j(h)), \varphi = 0 \text{ on } \partial\mathcal{M}_i(h)\backslash\mathcal{S}_j, \right.$$
$$\left. \|\varphi\|_{L^2(\mathcal{M}_j(h))} = 1\right\}.$$

We let

$$\widetilde{\lambda}_1(\mathcal{M}_0(h)) = \min\{\lambda_1(\mathcal{M}_1(h)), \cdots, \lambda_1(\mathcal{M}_\ell(h)), \mu_1(\mathcal{M}_{\ell+1}(h)), \cdots, \mu_1(\mathcal{M}_N(h))\}.$$

Then we have

$$\overline{\lambda}(h) = \widetilde{\lambda}_1(\mathcal{M}_0(h)).$$

More precisely, we can prove the following existence and nonexistence theorem of positive solutions of problem (**) (cf. [Cr, Théorème 6], [O, Theorem 3]):

**Theorem.** *Assume that:*

($\alpha$) $h \ge 0$ *in* $M$.

($\eta$) *The open set* $\mathcal{M}_0(h)$ *consists of a finite number of connected components* $\mathcal{M}_i(h)$, $1 \le i \le \ell$, *with smooth boundary which are bounded away from* $\partial M$, *and of a finite number of connected components* $\mathcal{M}_j(h)$, $\ell+1 \le j \le N$, *with smooth boundary such that each closure* $\overline{\mathcal{M}_j(h)}$ *is a neighborhood of some connected component* $\mathcal{S}_j$ *of* $\partial M$.

($\beta$) $a \ge 0$ *on* $\partial M\backslash\mathcal{S}_j$, *and* $a = 0$ *on* $\mathcal{S}_j$, $\ell+1 \le j \le N$.

*Then we have the following (cf. Figure 2):*

*(i) For any* $0 < \lambda < \widetilde{\lambda}_1(\mathcal{M}_0(h))$, *there exists a strictly positive solution* $u(\lambda)$ *of problem* (**).

*(ii) For any* $\lambda \ge \widetilde{\lambda}_1(\mathcal{M}_0(h))$, *there exists no positive solution of problem* (**).

*Furthermore, we have*

$$\lim_{\lambda \to \widetilde{\lambda}_1(\mathcal{M}_0(h))} \|u(\lambda)\|_{L^2(M)} = +\infty.$$

Our Main Theorem is an immediate consequence of this theorem.

## Bibliography

[Ag] Agmon, S., *Lectures on elliptic boundary value problems*, Van Nostrand, Princeton, 1965.

[Au] Aubin, T., *Equations différentielles non linéaire et problèmes de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. **55** (1976), 269–296.

[Cl] Chavel, I., *Eigenvalues in Riemannian geometry*, Academic Press, Orlando London, 1984.

[Cr] Cherrier, P., *Problèmes de Neumann non linéaires sur les variétés Riemanniennes*, J. Functional Analysis **57** (1984), 154–206.

[CR] Crandall, M. G. and P. H. Rabinowitz, *Bifurcation from simple eigenvalues*, Jour. Functional Anal. **8** (1971), 321–340.

[E] Escobar, J., *The Yamabe problem on manifolds with boundary*, J. Differential Geometry **35** (1992), 21–84.

[K] Kazdan, J. L., *Prescribing the curvature of a Riemannian manifold*, CBMS Regional Conference Series in Mathematics, No. 57, American Mathematical Society, Rhode Island, 1985.

[KW] Kazdan, J. L. and F. W. Warner, *Scalar curvature and conformal deformation of Riemannian structure*, J. Differential Geometry **10** (1975), 113–134.

[LP] Lee, J. and T. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. **17** (1987), 37–91.

[O] Ouyang, T. C., *On the positive solutions of semilinear equations* $\Delta u + \lambda u - hu^p = 0$ *on the compact manifolds*, Trans. Amer. Math. Soc. **331** (1992), 503–527.

[S] Schoen, R., *Conformal deformation of a Riemannian metric to a constant scalar curvature*, J. Differential Geometry **20** (1984), 479–495.

[Ta] Taira, K., *On the existence of Feller semigroups with boundary conditions*, Memoirs of the American Mathematical Society. No. 475, American Mathematica Society, Rhode Island, 1992.

[Tr] Trudinger, N. S., *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa **22** (1968), 265–274.

[Y] Yamabe, H., *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. **12** (1960), 21–37.