数理解析研究所講究録
第 838 巻 1993 年 35-39



# A NOTE ON SPACES OF RATIONAL FUNCTIONS

Martin Guest
Andrzej Kozlowski
Mitutaka Murayama
Kohhei Yamaguchi

The University of Rochester
Toyama International University
Tokyo Institute of Technology
The University of Electro-Communications

## §1. Introduction

Spaces of holomorphic maps between complex manifolds have played an important role in topology and mathematical physics. In this note, we shall consider the topology of spaces of holomorphic self-maps of the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ and announce the main results of [GKMY].

For a non-negative integer $d$, let $\mathrm{Hol}_d$ be the space of all holomorphic maps from $S^2$ to $S^2$ of degree $d$. Let $\mathrm{Hol}_d^*$ be the subspace of $\mathrm{Hol}_d$ consisting of all maps which preserve a basepoint of $S^2$. The corresponding space of continuous maps of degree $d$ is denoted by $\mathrm{Map}_d$, and the subspace of based maps by $\mathrm{Map}_d^*$. We call the map $f : X \to Y$ a *homotopy equivalence up to dimension* $d$ if the induced homomorphism $f_* : \pi_j(X) \to \pi_j(Y)$ is bijective for all $j < d$ and surjective for $j = d$. It is an elementary fact that $\mathrm{Hol}_d$ and $\mathrm{Hol}_d^*$ are connected spaces. For $d = 1$, it is easy to see that $\mathrm{Hol}_1 \cong PSL_2(\mathbb{C})$ and $\mathrm{Hol}_1^* \cong \mathbb{C}^* \times \mathbb{C}$. The following general results were obtained by Epshtein, J.D.S Jones and Segal:

**Theorem 0** ([Ep], [Se]). *Let $d$ be a positive integer.*
*(1)* $\pi_1(\mathrm{Hol}_d) = \mathbb{Z}/2d$.
*(2)* $\pi_1(\mathrm{Hol}_d^*) = \mathbb{Z}$.
*(3) The natural inclusion maps*

$$i_d : \mathrm{Hol}_d^* \to \mathrm{Map}_d^* \text{ and } j_d : \mathrm{Hol}_d \to \mathrm{Map}_d$$

*are homotopy equivalences up to dimension $d$.* □

The stable homotopy type of $\mathrm{Hol}_d^*$ was studied in [CCMM] and in this note we shall extend the above result by determing some further homotopy groups of the spaces $\mathrm{Hol}_d$ and $\mathrm{Hol}_d^*$. Our main results are as follows:

**Theorem 1.**

(1) *For $k \geq 2$,*

$$\pi_k(\mathrm{Hol}_d) = \begin{cases} \pi_k(S^3) & d = 1 \\ \pi_k(S^3) \oplus \pi_k(S^2) & d = 2 \\ \mathbb{Z}/2 & d \geq 3, k = 2 \end{cases}$$

(2) *If $k \geq 3$ and $d \geq 3$, then $\pi_k(\mathrm{Hol}_d) = \pi_k(\mathrm{Hol}_d^*) \oplus \pi_k(S^3)$.*

(3) *In particular, if $d > k \geq 3$, then $\pi_k(\mathrm{Hol}_d) = \pi_{k+2}(S^2) \oplus \pi_k(S^3)$.*

**Theorem 2.** *The space $\mathrm{Hol}_2$ may be identified with a homogeneous space of the form $(SL_2(\mathbb{C}) \times SL_2(\mathbb{C}))/H$, where $H$ is isomorphic to $\mathbb{C}^* \rtimes \mathbb{Z}/4$. In this semi-direct product, the action of $\mathbb{Z}/4 =< \sigma : \sigma^4 = 1 >$ is given by $\sigma \cdot \alpha = \alpha^{-1}$ for $\alpha \in \mathbb{C}^*$. In particular, $\mathrm{Hol}_2$ is homotopy equivalent to $(S^3 \times S^3)/(S^1 \rtimes \mathbb{Z}/4)$.*

**Theorem 3.**

(1) *The universal cover of $\mathrm{Hol}_2^*$ is homotopy equivalent to $S^2$.*

(2) *The universal cover of $\mathrm{Hol}_2$ may be identified with a homogeneous space of the form $(SL_2(\mathbb{C}) \times SL_2(\mathbb{C}))/D$, where $D$ is isomorphic to $\mathbb{C}^*$. In particular, it is homotopy equivalent to $S^3 \times S^2$.*

In this note, we shall discuss only Theorem 2 and (1) of Theorem 3, referring to [GKMY] for the remaining results.

*Acknowledgements:* The first, second and fourth authors are indebted to the Mathematics Department of Tokyo Institute of Technology for its hospitality. A gap in an earlier version of the proof of Theorem 3 was filled thanks to a suggestion of Professor A. Kono. The authors are grateful to Professors Cohen and Kono for their kind assistance.

## §2. Sketch proof of Theorem 2

From now on, we identify $\mathrm{Hol}_d$ with the space of functions $f = p_1/p_2$, where $p_1, p_2$ are coprime polynomials such that $\max\{\deg(p_1), \deg(p_2)\} = d$. The group $\mathrm{Hol}_1$ acts on $\mathrm{Hol}_d$ by pre- and post-compositions: for $(A, B) \in \mathrm{Hol}_1 \times \mathrm{Hol}_1$ and $f \in \mathrm{Hol}_d$ we have

$$(A, B) \cdot f(z) = A(f(B^{-1}(z))).$$

The following proposition is well known:

**Proposition 2.1.** *The group $\mathrm{Hol}_1 \times \mathrm{Hol}_1$ acts transitively on $\mathrm{Hol}_2$.* □

It is well known that the map

$$SL_2(\mathbb{C}) \to \mathrm{Hol}_1, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az + b}{cz + d},$$

is a double covering and induces an isomorphism $PSL_2(\mathbb{C}) \cong \mathrm{Hol}_1$ of Lie groups. Thus the group $G = SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ acts (transitively) on $\mathrm{Hol}_2$.

**Lemma 2.2.** *Let $H$ denote the isotropy subgroup of $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ at $z^2 \in \mathrm{Hol}_2$. Then*

$$H=\left\{\left(\pm\begin{pmatrix}\alpha^2 & 0\\ 0 & \alpha^{-2}\end{pmatrix},\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\right),\left(\pm\begin{pmatrix}0 & i\alpha^2\\ i\alpha^{-2} & 0\end{pmatrix},\begin{pmatrix}0 & \alpha\\ -\alpha^{-1} & 0\end{pmatrix}\right):\alpha\in\mathbb{C}^*\right\}.$$

*Proof.* This follows by direct calculation. □

It is also easy to establish:

**Lemma 2.3.** *Let $K = \mathbb{C}^* \rtimes \mathbb{Z}/4$ be the group defined by the action of $\mathbb{Z}/4 =< \sigma : \sigma^4 = 1 >$ on $\mathbb{C}^*$ by $\sigma \cdot \alpha = \alpha^{-1}$ for $\alpha \in \mathbb{C}^*$. Then $H$ and $K$ are isomorphic Lie groups.* □

*Proof of Theorem 2.* The first part of the Theorem follows easily from Proposition 2.1, Lemma 2.2 and Lemma 2.3. The inclusion map of the maximal compact subgroup $SU(2) = S^3$ of $SL_2(\mathbb{C})$ induces the homotopy equivalence $(S^3 \times S^3)/(S^1 \rtimes \mathbb{Z}/4) \simeq \mathrm{Hol}_2$. □

## §3. The universal cover of $\mathrm{Hol}_2^*$

In this section, we shall show that the universal cover of $\mathrm{Hol}_2^*$ is homotopy equivalent to $S^2$.

**Proposition 3.1** ([CS]). *There is a fibration $S^1 \longrightarrow \mathrm{Hol}_2^* \xrightarrow{\pi} \mathbb{R}P^2$.*

*Remark.* R.Cohen and D.Shimamoto ([CS]) deduce this from results of Donaldson ([D]) and Atiyah and Hitchin ([AH]) on monopoles. Although we give an elementary proof using the homogeneous structure of $\mathrm{Hol}_2$ in [GKMY], we shall give here an alternative direct proof.

*Proof.* We may identify $\mathrm{Hol}_2^*$ with the space of all holomorphic self maps $f$ of $S^2 = \mathbb{C} \cup \{\infty\}$ of degree 2 with basepoint condition $f(\infty) = 0$. Then any $f \in \mathrm{Hol}_2^*$ is of the form

$$f(z) = (az + b)/(z^2 + cz + d)$$

where the polynomials $az+b$ and $z^2+cz+d$ are coprime, $(a,b) \neq (0,0)$ and $a,b,c,d \in \mathbb{C}$.

For each pair $(U,V)$ of subspaces of $S^2$, let $Q_{m,n}(U,V)$ be the space of all disjoint positive divisors of $Sp^m(U) \times Sp^n(V)$,

$$Q_{m,n}(U,V) = \{(\xi,\eta) \in Sp^m(U) \times Sp^n(V) : \xi \cap \eta = \emptyset\},$$

where $Sp^k(X)$ denotes the $k$-th symmetric product of $X$, $Sp^k(X) = X^k/\Sigma_k$. ($\Sigma_k$ is the symmetric group on $k$ letters.)

The action of $\mathbb{C}^*$ on $S^2 = \mathbb{C} \cup \{\infty\}$ (by multiplication) induces a free action on $\mathrm{Hol}_2^*$ and the quotient space $\mathrm{Hol}_2^* / \mathbb{C}^*$ may be identified with $Q_{1,2}(S^2, \mathbb{C})$. It suffices to show that $Q_{1,2}(S^2, \mathbb{C}) \simeq \mathbb{R}P^2$.

Observe that

$$Q_{1,2}(S^2, \mathbb{C}) \simeq Q_{1,2}(S^2, D_-) = Q_{1,2}(\overline{D}_+, D_-) \cup Q_{1,2}(\overline{D}_-, D_-),$$

where $D_\pm$ are the open northern and southern hemispheres of $S^2 = \{z \in \mathbb{R}^3 : \|z\| = 1\}$.

The map $u : Q_{1,2}(\overline{D}_+, D_-) \to Q_{1,2}(\overline{D}_+, \{0\}) \cong \overline{D}_+$ given by

$$(\alpha, \beta_1 + \beta_2) \mapsto (\alpha, 0 + 0)$$

is a homotopy equivalence.

The map $v : Q_{1,2}(\overline{D}_-, D_-) \to \mathbb{C}^*$ given by

$$(\alpha, \beta_1 + \beta_2) \mapsto (\alpha - \beta_1)(\alpha - \beta_2)$$

is also a homotopy equivalence. Regarding the intersection

$$Q_{1,2}(\overline{D}_+, D_-) \cap Q_{1,2}(\overline{D}_-, D_-) = Q_{1,2}(S^1, D_-),$$

we have the homotopy commutative diagram

$$\begin{tikzcd} Q_{1,2}(S^1, D_-) \arrow[r, "="] \arrow[d, "u"'] & Q_{1,2}(S^1, D_-) \arrow[d, "v"'] \\ Q_{1,2}(S^1, \{0\}) \arrow[r, "f"] & \mathbb{C}^* \end{tikzcd}$$

It follows from the definition of the map $v$ that the bottom map $f$ is given by the map

$$Q_{1,2}(S^1, \{0\}) \cong S^1 \to \mathbb{C}^*; S^1 \ni z \mapsto z^2 \in \mathbb{C}^*.$$

Hence

$$Q_{1,2}(S^2, \mathbb{C}) \simeq Q_{1,2}(S^2, D_-) \simeq \mathbb{C}^* \cup_{f \circ u} Q_{1,2}(\overline{D}_+, D_-) \simeq S^1 \cup_2 e^2 \simeq \mathbb{R}P^2. \quad \square$$

*Proof of (1) of Theorem 3.*

Consider the fibration $S^1 \to \mathrm{Hol}_2^* \xrightarrow{\pi} \mathbb{R}P^2$. Let $p : S^2 \to \mathbb{R}P^2$ and $q : \tilde{\mathrm{Hol}}_2^* \to \mathrm{Hol}_2^*$ be the universal coverings. Since $\tilde{\mathrm{Hol}}_2^*$ is simply connected, there is a lift $\theta : \tilde{\mathrm{Hol}}_2^* \to S^2$ such that $p \circ \theta = \pi \circ q$. It follows from diagram chasing that $\theta_* : \pi_j(\tilde{\mathrm{Hol}}_2^*) \to \pi_j(S^2)$ is an isomorphism for all $j \geq 2$. Since both spaces are simply connected, $\theta$ is a homotopy equivalence. $\square$

*Remark.* Using Theorem 3, we can prove that the $C_2$-operad structure on $\mathrm{Hol}^* = \amalg_{d\geq 0}\mathrm{Hol}_d^*$ given by [BM] is not compatible with that on $\Omega^2 S^2$ up to homotopy. See [GKMY] for the details. $\square$

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