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# On the Igusa's local zeta functions for curves

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## §0. Introduction

Let $K$ be a nonarchimedean local field of characteristic zero with its ring of integers $O$. let $\pi O$ be the unique maximal ideal of $O$, and let $q$ be the cardinality of the residue field $O/\pi O$.

For a polynomial $f(x)$ in $K[x]$, where $x = (x^1, x^2, \cdots, x^n)$, the Igusa's local zeta function of $f$ is defined as

$$I(f) = \int_{O^n} |f(x)|^s |dx|.$$

where $|\ \ |$ is the usual absolute value on $K$ and $dx$ is the usual Haar measure on $K$ such that the measure of $O$ is $1$.

As a function of $s \in \mathbb{C}$, it is known [I] that $I(f)$ is holomorphic for $\mathrm{Re}(s) > 0$ with a meromorphic continuation to the entire complex plane, and it is rational in $t = q^{-s}$.

Let $f(x, y) \in O[x, y]$, and let $\bar{f}$ be the natural projection of $f$ under $O \longrightarrow O/\pi O \cong \mathbb{F}_q$. We assume that the curve defined by $\bar{f}(x, y) = 0$ has its only singularity at $(0, 0)$. By the resolution process, Meuser [M] showed that there is exactly one simple pole comes from each "characteristic exponent" of the puiseux expansion of $f$. Characteristic exponents were first considered in connection with the above zeta function in [I].

Aiming at finding an algorithm to compute the local zeta funcitons named after him, Igusa introduced the p–adic stationary phase formula (SPF in short), which turns out to be a very powerful tool, as our result shows. A clarification of the relation between the arithmetic desingularization of the curve $f(x, y) = 0$ and an algorithm via SPF to compute $I(f)$ was asked by Oesterlé. In fact, in general, there is a correspondence between the desingularization by monoidal transformation and the computation of Igusa's local zeta functions by SPF.

In this paper, we shall carry out this program for $n = 2$. We assume that in the puiseux expansion of $f(x, y) = 0$ only characteristic exponents appears and they are with coefficients 1. An explicit formula for the Igusa's local zeta function is then obtained in terms of these characteristic exponents. It will become evident in our algorithm that those terms with noncharacteristic exponents (with integer coefficients) make no comtribution to our zeta function.

## §1. The tool: SPF

### Lemma 1. (SPF)

Let $f(x) \in O[x] = O[x^1, x^2, \cdots, x^n]$. Under the projection $O \longrightarrow O/\pi O \cong \mathbb{F}_q$, $f(x)$ is mapped to $\bar{f}(x) \in \mathbb{F}_q[x]$. Then

$$
\begin{aligned}
I(f) &= \int_{O^n} |f(x)|^s |dx| \\
&= 1 - q^{-n}N^{\#} + q^{-n}(N^{\#} - S^{\#})t(1-q^{-1})(1-q^{-1}t)^{-1} \\
&\qquad + q^{-n} \sum_{\substack{x_0 \in O^n \bmod \pi \\ \bar{x}_0 \in S}} \int_{O^n} |f(x_0 + \pi x)|^s |dx| \qquad (1.1)
\end{aligned}
$$

where
$$N = \{\bar{x} \in \mathbb{F}_q^n \mid \bar{f}(\bar{x}) = 0\}$$

$$S = \{\bar{x} \in \mathbb{F}_q^n \mid \bar{f}(\bar{x}) = \nabla\bar{f}(\bar{x}) = 0\}$$

and $N^{\#}$ = card (N), $S^{\#}$ = card (S). (In case of emphasizing its dependance to f, we shall write $N(f)$, $N^{\#}(f)$ etc.). $t = q^{-s}$.

## §2. Igusa's local zeta funciton of $y^m - x^n$

For the computation of the Igusa' local zeta function, the most simplest case is the following.

**Example 1.** $f(x, y) = y^2-x^3$.

Let $I = I(f) = \int_{O^2} |y^2-x^3|^s \, |dx| \, |dy|$.

Apply SPF to I(f), since

$$N(f) = \{(u^2, u^3) \mid u \in \mathbb{F}_q\},$$

$$S(f) = \{(0, 0)\},$$

$$N^{\#}(f) = q.$$

We get

$$I = 1 - q^{-1} + q^{-1}(1-q^{-1})^2 t(1-q^{-1}t)^{-1} + q^2 \int_{O^2} |(\pi y)^2-(\pi x)^3|^s |dx| \, |dy|$$

where

$$\int_{O^2} |(\pi y)^2-(\pi x)^3|^s |dx| \, |dy| = t^2 \int_{O^2} |y^2-\pi x^3|^s \, |dx| \, |dy|.$$

Introduce the following notations

$$f_{1,1} = y^2 - \pi x^3$$

$$I_{1,1} = I(f_{1,1})$$

then $$I = 1 - q^{-1} + q^{-1}(1-q^{-1})^2 t(1-q^{-1}t)^{-1} + q^{-2} t^2 I_{1,1}. \tag{2.1}$$

Apply SPF once again to $I(f_{1,1})$, since

$$N(f_{1,1}) = S(f_{1,1}) = \{(\xi, 0) \mid \xi \in \mathbb{F}_q\}$$

we have

$$I_{1,1} = 1 - q^{-1} + q^{-2} \sum_{x_0 \in O/\pi} \int_{O^2} |(\pi y)^2 - \pi(x_0+\pi x)^3|^s |dx|\ |dy|$$

in which the summation equals to

$$t \cdot q \cdot \int_{O^2} |\pi y^2 - x^3|^s |dx|\,|dy|$$

(here we make te change of variable $x \mapsto x_0+\pi x$).

Hence

$$I_{1,1} = 1 - q^{-1} + q^{-1} t\, I_{1,2} \tag{2.2}$$

in which

$$I_{1,2} = I(f_{1,2}), \qquad f_{1,2} = \pi y^2 - x^3.$$

For similar reason, we have

$$I_{1,2} = 1 - q^{-1} + q^{-1} t\, I_{2,2} \tag{2.3}$$

$$I_{2,2} = 1 - q^{-1} + q^{-1} t^2\, I_{2,3} \tag{2.4}$$

in which

$$I_{2,2} = I(f_{2,2}), \qquad f_{2,2} = y^2 - \pi^2 x^3$$

$$I_{2,3} = I, \qquad f_{2,3} = f$$

summarize (2.2), (2.3) and (2.4), we get

$$I_{1,1} = (1-q^{-1})(1+q^{-1}t+q^{-2}t^2) + q^{-3}t^4 I \tag{2.5}$$

Compare (2.1) with (2.5), we obtain the local zeta function of $y^2-x^3$ as

$$I(y^2-x^3) = (1-q^{-5}t^6)^{-1}(1-q^{-1})[1+q^{-2}t^2+q^{-3}t^3+q^{-4}t^4+q^{-1}(1-q^{-1})t(1-q^{-1}t)^{-1}].$$

In the general case, let $f(x, y) = y^m-x^n$, where $m$ and $n$ are coprime and $1 < m < n$. Proceed as in the Example 1, let $f_{0,0} = f$ and

$$I_{0,0} = \int_{O^2} |f_{0,0}(x, y)|^s dxdy$$

then $I = I(f) = I_{0,0}$. Apply SPF to $I_{0,0}$ once, since $N^{\#}(f) = q$ and $S = \{(0, 0)\}$ we get

$$I_{0,0} = 1 - q^{-1} + (1-q^{-1})^2 q^{-1} t(1-q^{-1}t)^{-1} + q^{-2}\int_{O^2} |f_{0,0}(\pi x, \pi y)|^s |dx| \, |dy|.$$

Let $f_{0,0}(\pi x, \pi y) = \pi^m f_{1,1}(x, y)$, then we have

## Proposition 2.0.

$$I = I_{0,0} = 1-q^{-1} + (1-q^{-1})^2 q^{-1} t(1-q^{-1}t)^{-1} + q^{-2} t^m I_{1,1} \tag{2.6}$$

where $\quad I_{1,1} = I(f_{1,1}),\ \ f_{1,1}(x, y) = y^m - \pi^{n-m} x^n.$

For $1 \le i \le m$, $[(i-1)n/m]+1 \le j \le [in/m] + 1$, define

$$f_{i,j}(x, y) = \begin{cases} y^m - \pi^{in-jm} x^n, & \text{if } j \le [in/m] \\ \pi^{(j+1)m-in}\, y^m - x^n, & \text{if } j = [in/m]+1. \end{cases} \tag{2.7}$$

$$I_{i,j} = I(f_{i,j})$$

then we have

## Proposition 2.1.

$$I_{i,j} = \begin{cases} 1 - q^{-1} + q^{-1}t^{m} \cdot I_{i,\,j+1} & \text{if } j < [in/m] \\ 1 - q^{-1} + q^{-1}t^{r_i} I_{i},\, j+1, & \text{if } j = [in/m] \\ 1 - q^{-1} + q^{-1}t^{m-r_i} I_{i+1\,,\,j} & \text{if } j = [in/m]+1 \end{cases} \tag{2.8}$$

where $in = [in/m] \cdot m + r_i, \quad 0 < r_i < m.$

## Proposition 2.5.

$$I_{1,1} = (1-q^{-1})(q^{-1}t^{m})^{-1} \cdot P(t) + q^{-(n+m-2)}t^{m(n-1)}I_{m,n} \tag{2.13}$$

where

$$P(t) = \sum_{i=1}^{m-1} q^{-[in/m]-i}t^{in} + \sum_{j=1}^{n-1} q^{-[jm/n]-j}t^{jm}. \tag{2.14}$$

## Theorem 1. (Igusa's local zeta functions for $y^m - x^n$)

Let $I$ be the Igusa's local zeta function for $f(x, y) = y^m - x^n$, where $n > m > 1$ and they are coprime. Then

(i) $$I = (1-q^{-(m+n)}t^{mn})^{-1} \cdot (1-q^{-1})\{1+q^{-1}(1-q^{-1})t(1-q^{-1}t)^{-1} + q^{-1}P(t)\}. \tag{2.15}$$

(ii) $$I_{1,1} = (1-q^{-(m+n)}t^{mn})^{-1}(1-q^{-1})\{(q^{-1}t^{m})^{-1}P(t) + q^{-(m+n-2)}t^{(n-1)m} + (1-q^{-1})q^{-1}t(1-q^{-1}t)^{-1}q^{-(m+n-2)}t^{(n-1)m}\} \tag{2.16}$$

where the polynomial $P(t)$ is given by (2.13) and $I_{1,1}$ is defined in the proposition 2.0.

## §3. The general setup of the computing algorithm

Let $f(x, y) \in K[x, y]$. We may assume that $y$ is expanded in the puiseux series in the ascending exponents:

$$y = \sum_{i=1}^{k_0} a_{0,i}\, x^i + \sum_{i=0}^{k_1} a_{1,i}\, x^{(n_1+i)/m_1} + \cdots$$

$$+ \sum_{i=0}^{k_{g-1}} a_{g-1,i}\, x^{(n_g+i)/m_1 m_2 \cdots m_{g-1}}$$

$$+ \sum_{i=1}^{\infty} a_{g,i}\, x^{(n_g+i)/m_1 m_2 \cdots m_g}$$

in which $m_i$ and $n_i$ are coprime integers and $n_i > m_i > 1$ and $a_{j,0} \neq 0$ for all $1 \leq j \leq g$. the corresponding $g$ exponents

$$n_1/m_1,\ n_2/m_1 m_2,\ \cdots,\ n_g/m_1 m_2 \cdots m_g$$

are called the "characteristic exponents" of the curve.

In the following sections we shall assume

$$a_{j,0} = 1 \quad \text{for all } j \text{ and } a_{j,i} = 0 \text{ for all } j, \text{ all } i \neq 0.$$

It will become evident in our algorithm which appears in the following sections that those non–characteristic terms (with integer coefficients) will have no contribution to the integral.

**Notations.** For $1 \le i \le g$, $n_i$ and $m_i$ are coprime and $n_i > m_i \ge 2$.

Put $m_i' = \prod_{1 \le \lambda \le i} m_\lambda$, $\quad m_i'' = \prod_{1 < \lambda \le g} m_\lambda$, $\quad m = \prod_{1 \le \lambda \le g} m_\lambda = m_i' m_i''$

$\tilde{\xi} = \xi^{-1}$, for all natural number $\xi$.

$\ell_\lambda = n_\lambda - n_1 \tilde{m}_1 m_\lambda'$, $1 \le \lambda \le g$

(then $\ell_\lambda > 0$ for $\lambda > 1$, and $\ell_1 = 0$)

$\ell_\lambda^* = \ell_\lambda - \ell_{\lambda-1} m_\lambda$.

Let $y = \sum_{i=1}^{g} x^{n_i \tilde{m}_i'}$ be a puiseux series with fractional powers, and let

$$f(x, y) = \prod_{k \bmod m} \left[ y - \sum_{\lambda=1}^{g} \epsilon^{k n_\lambda m_\lambda''} x^{n_\lambda \tilde{m}_\lambda'} \right]$$

be the product of all conjugates of the puiseux series, where $\epsilon$ is a primitive root of unity of order $m$.

Let $f^{(0)} = f$, $\qquad I^{(0)} = I(f^0)$

$$I = I(f) = \int_{O^2} |f(x, y)|^s |dx|\ |dy|$$

We shall assume that (0, 0) is the only singularity for $\bar{f} = 0$ over $\mathbb{F}_q$. Then we have

## Main Theorem.

Let $g$ be a natural number. For $1 \le i \le g$, $n_i$ and $m_i$ are coprime and $n_i > m_i \ge 2$. Given a puisenx series

$$y = \sum_{i=1}^{g} x^{n_i(m_1 m_2 \cdots m_i)^{-1}}.$$

Let $f(x, y)$ be the product of all $m_1 m_2 \cdots m_g$ conjugates of $y - \sum_{i=1}^{g} x^{n_i}(m_1 m_2 \cdots m_i)^{-1}$. Let $I(f) = \int_{O^2} |f(x, y)|^s \, |dx| \, |dy|$, then we have

(i) $\quad I(f) = 1 - q^{-1} + (N_0-1)q^{-2}t(1-q^{-1})(1-q^{-1}t)^{-1} + q^{-2}t^{m}I_{1,1}^{(0)}$

(ii) $\quad I_{1,1}^{(0)} = (1-\tau)^{-1}(1-q^{-1})(q^{-1}t^{m})(q^{-1}t^{m})^{-1}[P(t^{m\tilde{m}_1}) + q\tau]$
$\quad\quad + (1-q^{-1})(q^{-1}t^{m})^{-1}\tau\cdot\Omega.$

(iii) $\Omega = \Omega^{(g)}$

$$= (1-q^{-1}) \sum_{\lambda=1}^{g-1} (1-\tau_\lambda)^{-1}(1-q^{-1}t^{\tilde{m}'_\lambda m})^{-1}(\tilde{m}_1 m'_\lambda)t^{\tilde{m}'_\lambda m}\cdot\tau^{-1}\tau_\lambda$$

$$- \sum_{\lambda=1}^{g-1} (1-\tau_{\lambda+1})^{-1}\{(1-q^{-1})\cdot q\cdot(\tilde{m}_1 m'_\lambda)(1-q^{-1}t^{\tilde{m}'_\lambda m})^{-1}\tau^{-1}\tau_\lambda\phi_{\lambda+1}$$

$$- (1-q^{-1})q\cdot(\tilde{m}_1 m'_\lambda)\tau^{-1}\tau_\lambda(\phi_{\lambda+1} - \tau_\lambda^{-1}\tau_{\lambda+1})$$

$$- (\tilde{m}_1 m'_\lambda)\tau^{-1}\tau_\lambda(\varphi_{\lambda+1} - \tau_\lambda^{-1}\tau_{\lambda+1})$$

$$- q\cdot(\tilde{m}_1 m'_\lambda)(1-m_{\lambda+1}q^{-1})\tau^{-1}\tau_{\lambda+1}\}$$

$$+ (1-\tau_g)^{-1}\cdot(\tilde{m}_1 m)(1-q^{-1})t(1-q^{-1}t)^{-1}\tau^{-1}\tau_g$$

where $N_0$ is the number of solutions to $\bar{f}(x, y) = 0$ over $\mathbf{F}_q$, $P(t)$ is defined by (2.14), $\{\tau_\lambda\}_{1\le\lambda\le g}$ are defined recursively by (5.3), $m'_\lambda = \prod_{1\le i\le\lambda} m_i$, $m = \prod_{1\le i\le g} m_i$, $\tilde{\xi} = \xi^{-1}$, $\phi_\lambda$ and $\phi_\lambda$ are defined respectively by (5.4) and (5.5).

## §8. Examples for $g=2$ and $g=3$

**Example 2.** $y = x^{3/2} + x^{9/4}$

$$f = (y-x^{3/2}-x^{9/4})(y-x^{3/2}+x^{9/4})(y+x^{3/2}-ix^{9/4})(y+x^{3/2}+ix^{9/4})$$
$$= (y^2-x^3)^2 - 4x^6y - x^9.$$

$$I(f) = 1 - q^{-1} + q^{-1}(1-q^{-1})^2t(1-q^{-1}t)^{-1} + q^{-2}t^4I^{(0)}_{1,1}$$
$$I^{(0)}_{1,1} = (1-q^{-5}t^{12})^{-1}(1-q^{-1})(1+q^{-1}q^2+q^{-2}t^4+q^{-3}t^8)$$
$$+ (1-q^{-1})q^{-4}t^8\cdot\Omega .$$

For $g = 2$, the general formula for $\Omega$ is

$$\Omega = (1-\tau)^{-1}\cdot(1-q^{-1})(1-q^{-1}t^{m_2})^{-1}t^{m_2} - (1-\tau_2)^{-1}\cdot\{(1-q^{-1})(1-q^{-1}t^{m_2})^{-1}\cdot q\phi_2$$
$$- (1-q^{-1})q(\phi_2-\tau^{-1}\tau_2) - (\varphi_2-\tau^{-1}\tau_2) - (1-m_2q^{-1})q\tau^{-1}\tau_2\}$$
$$+ (1-\tau_2)^{-1}\cdot m_2(1-q^{-1})t(1-q^{-1}t)^{-1}\cdot\tau^{-1}\tau_2.$$

In Example 2,

$$\ell_2 = \ell_2^* = 3$$
$$\tau = \tau_1 = q^{-5}t^{12},\ \ \tau_2 = (q^{-5}t^{12})^2(q^{-1}t^2)^3 = q^{-13}t^{30}$$
$$\phi_2 = q^{-1}t^2 + q^{-8}t^8,\ \ \varphi_2 = q^{-1}t^3 + q^{-8}t^8$$
$$\Omega = (1-q^{-5}t^{12})^{-1}(1-q^{-1})(1-q^{-1}t^2)^{-1}t^2$$
$$- (1-q^{-13}t^{30})^{-1}\cdot\{(1-q^{-1})(1-q^{-1}t^2)^{-1}(t^2+q^{-7}t^{18})$$
$$- (1-q^{-1})t^2 - q^{-1}t^3 - (1-2q^{-1})q^{-7}t^{18}\}$$
$$+ (1-q^{-13}t^{30})^{-1}\cdot 2(1-q^{-1})(1-q^{-1}t)^{-1}\cdot q^{-8}t^{19}.$$

**Example 3.** $y = x^{3/2} + x^{7/4} + x^{17/8}$

$$I = 1-q^{-1}+(N_0-1)tq^{-2}(1-q^{-1})(1-q^{-1}t)^{-1}+q^{-2}t^{8}\, I^{(0)}_{1,1}$$

$$I^{(0)}_{1,1} = (1-q^{-5}t^{24})^{-1}\cdot(1-q^{-1})(1+q^{-1}t^{4}+q^{-2}t^{8}+q^{-3}t^{16}) + (1-q^{-1})q^{-4}t^{16}\Omega$$

For $g = 3$, the general formula for $\Omega$ is

$$\begin{aligned}
\Omega = {} & (1-q^{-1})(1-q^{-1}t^{m_2m_3})^{-1}t^{m_2m_3}\cdot(1-\tau_1)^{-1} \\
& + (1-\tau_2)^{-1}\{(1-q^{-1})(1-q^{-1}t^{m_3})^{-1}m_2\cdot t^{m_3}\cdot\tau^{-1}\tau_2 \\
& \qquad - (1-q^{-1})q(1-q^{-1}t^{m_2m_3})^{-1}\phi_2 + (1-q^{-1})q\cdot(\phi_2-\tau^{-1}\tau_2) \\
& \qquad + (\varphi_2 - \tau^{-1}\tau_2) + q\cdot(1-m_2q^{-1})\tau^{-1}\tau_2\} \\
& - (1-\tau_3)^{-1}\cdot\{(1-q^{-1})q\cdot m_2\cdot(1-q^{-1}t^{m_3})^{-1}\tau^{-1}\tau_2\phi_3 \\
& \qquad - (1-q^{-1})q\, m_2\, \tau^{-1}\tau_2\cdot(\phi_3-\tau_2^{-1}\tau_3) - m_2\cdot\tau^{-1}\tau_2(\varphi_3-\tau_2^{-1}\tau_3) \\
& \qquad - qm_2(1-m_3q^{-1})\tau^{-1}\tau_3\} \\
& + (1-\tau_3)^{-1}(m_2m_3)(1-q^{-1})t(1-q^{-1}t)^{-1}\tau^{-1}\tau_3.
\end{aligned}$$

In Example 3,

$\ell_2 = 1 = \ell_2^*,\ \ \ell_3 = 5,\ \ \ell_3^* = 3$

$\tau = \tau_1 = q^{-5}t^{24},\ \ \tau_2 = (q^{-5}t^{24})^2(q^{-1}t^4) = q^{-11}t^{52}$

$\tau_3 = (q^{-11}t^{52})^2(q^{-1}t^2)^3 = q^{-25}t^{110}$

$\tau^{-1}\tau_2 = q^{-6}t^{28},\ \ \tau_2^{-1}\tau_3 = q^{-14}t^{58},\ \ \tau^{-1}\tau_3 = q^{-20}t^{86}$

$\phi_2 = 1 + q^{-6}t^{28},\ \ \phi_3 = q^{-1}t^2 + q^{-14}t^{58}$

$$\psi_2 = t^2 + q^{-6}t^{28}, \ \psi_3 = q^{-1}t^3 + q^{-14}t^{58}$$

$$\begin{aligned}\Omega = {} & (1-q^{-5}t^{24})^{-1}(1-q^{-1})(1-q^{-1}t^4)^{-1}t^4 \\ & + (1-q^{-11}t^{52})^{-1}\{(1-q^{-1}t^2)^{-1}\cdot 2(1-q^{-1})q^{-6}t^{30} \\ & - (1-q^{-1}t^4)^{-1}\cdot(1-q^{-1})\cdot q\cdot(1+q^{-6}t^{28}) \\ & \quad + (1-q^{-1})q + t^2 + (1-2q^{-1})q^{-5}t^{28}\} \\ & - (1-q^{-25}t^{110})^{-1}\cdot\{(1-q^{-1}t^2)^{-1}\cdot 2(1-q^{-1})\cdot(q^{-6}t^{30}+q^{-19}t^{86}) \\ & \quad - (1-q^{-1})\cdot 2q^{-6}t^{30} - 2q^{-7}t^{31} - 2(1-2q^{-1})q^{-19}t^{86}\} \\ & + (1-q^{-25}t^{110})^{-1}\cdot(1-q^{-1}t)^{-1}\cdot 4(1-q^{-1})\cdot q^{-20}t^{87}\end{aligned}$$

# References


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