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COMPLETELY INTEGRABLE QUANTUM SYSTEMS WITH COORDINATE SYMMETRIES AND HYPERGEOMETRIC EQUATIONS

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1. Introduction

The results in this article are a joint work with Hiroko Sekiguchi (Dept. of Mathematical Sciences, Univ. of Tokyo) and are given in [OS] with their proofs.

In analytical dynamics the motion of particles is described by Hamilton's canonical equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad \text{for } i = 1, \dots, n.$$

Here q_i are generalized coordinates and p_i are generalized momentums. Hamiltonian H is the total energy of this system and typically

$$H(p,q) = \frac{1}{2}p^2 + U(q),$$

where $\frac{1}{2}p^2 = \frac{1}{2}(p_1^2 + \cdots + p_n^2)$ is the energy of motion and $U(q)$ is the potential energy of the system.

In general any function $h(p,q)$ of (p,q) satisfies

$$\frac{dh}{dt} = \{H, h\}$$

with the Poisson bracket

$$\{f,g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i} \right).$$

A function $h(p,q)$ is called an integral of the system if

$$\{H, h\} = 0.$$

If there exist n functionally independent integrals $h_1 = H$, $h_2, \dots, h_n$ satisfying $\{h_i, h_j\} = 0$, Hamilton's canonical equations are transformed into trivial equations under the canonical coordinate system $(h_1, \dots, h_n, g_1, \dots, g_n)$ and we can easily analyze the motion of articles. The local existence of these functions is reduced to

classical mechanics. The system or H is called completely integrable if there exist globally these functions $h_1, \ldots, h_n$ in a suitable function space.

Replacing functions by differential operators and the Poisson bracket by the commutator of the operators, we have quantum systems. It is natural to replace p_i and q_j by $\partial_i = \frac{\partial}{\partial x_i}$ and x_j, respectively, because of the canonical commutation relations $\{p_i, p_j\} = \{q_i, q_j\} = 0$ and $\{p_i, q_j\} = \delta_{ij}$. We consider the differential operator

$$P = \partial_1^2 + \cdots + \partial_n^2 + V(x_1, \ldots, x_n)$$

corresponding to the Hamiltonian. We call the quantum system defined by P is completely integrable if there exist n algebraically independent differential operators $P_1 = P$ and $P_2, \ldots, P_n$ with $[P_i, P_j] = 0$ for $i, j = 1, \ldots, n$. Then our problem is to determine the potential function $V(x)$ such that the system is completely integrable.

Many classical completely integrable systems are related to Lie algebras and the integrability is clarified by the structure of the Lie algebras ([OP]). Because of this structure the potential function $V(x)$ has some symmetry and in fact $V(x)$ are usually symmetric functions of $(x_1, \ldots, x_n)$. Here in some cases the orbits of motions are described by the Lie group actions on suitable homogeneous spaces.

The systems of differential equations satisfied by zonal spherical functions on symmetric spaces give examples of completely integrable quantum systems ([HC]). In this case the potential function $V(x)$ has some parameters m_α which take special integers determined by the dimensions of the root spaces for the symmetric space. Jiro Sekiguchi [Sj] proved the complete integrability for general complex parameters m_α when the root system is of type A_n and Heckman-Opdam [H1], [H2], [HO], [Op1], [Op2] proved it in general case. In these cases the commuting differential operators $P_1, \ldots, P_n$ are invariant under the action of the Weyl group W of the root system. Moreover the principal symbols $\sigma(P_1), \ldots, \sigma(P_n)$ do not depend on x and generate W-invariants of $\mathbb{C}[\xi_1, \ldots, \xi_n]$. For example, if the root system is of type A_{n-1}, the actions of W are identified with the permutations of the coordinates $x_1, \ldots, x_n$.

Let W be a classical Weyl group naturally acting on $\mathbb{R}^n$. In this article we shall study the potential function $V(x)$ which allows the W-invariant commuting differential operators $P_1, \ldots, P_n$ with $P_1 = P$ such that $\sigma(P_1), \ldots \sigma(P_n)$ do not depend on x and generate the W-invariants of $\mathbb{C}[\xi]$. We assume that there exist a connected open neighborhood Ω of the origin in $\mathbb{C}^n$ such that $V(x)$ is holomorphically extended on Ω', where $\Omega \setminus \Omega'$ is a proper analytic subset of Ω.

2. Type A_{n-1}

In this section we suppose the Weyl group W is of type A_{n-1} with $n > 2$. Then W is identified with the permutation group of the coordinates $x_1, \ldots, x_n$ of $\mathbb{R}^n$. Then our problem is to study the following system.

Let $\Delta_1, \ldots, \Delta_n$ be W-invariant differential operators of the form

$$\Delta_1 = \partial_1 + \cdots + \partial_n,$$

$$\Delta_2 = \sum_{1 \le i < j \le n} \partial_i \partial_j + R(x),$$

$$\Delta_k = \sum_{1 \le i_1 < \cdots < i_k \le n} \partial_{i_1} \cdots \partial_{i_k} + R_k(x, \partial) \quad \text{for } 3 \le k \le n$$

such that

$$[\Delta_i, \Delta_j] = 0 \quad \text{for } 1 \le i < j \le n,$$
$$\operatorname{ord} R_k(x, \partial) < k \quad \text{for } 3 \le k \le n.$$

Here the operator P in §1 corresponds to $\Delta_1^2 - 2\Delta_2$.

Then we have the following

Theorem 2.1. *There exist an even holomorphic function $u(t)$ defined for $0 < |t| << 1$ such that*

$$R(x) = \sum_{1 \le i < j \le n} u(x_i - x_j). \tag{2.1}$$

Theorem 2.2. *The commuting algebra $\mathbb{C}[\Delta_1, \ldots, \Delta_n]$ is uniquely determined by $R(x)$ if $\operatorname{ord} R_3(x, \partial) < 2$.*

Remark. The assumption $\operatorname{ord} R_3(x, \partial) < 2$ is removed by K. Taniguchi in Theorem 2.2 except for the trivial case where $R(x)$ is constant.

Theorem 2.3. *Under the notation in Theorem 2.1 there exist complex numbers A_0, A_1, ω_1 and ω_2 such that*

$$u(t) = A_1 \wp(t|2\omega_1, 2\omega_2) + A_0. \tag{2.2}$$

Here $\wp(t|2\omega_1, 2\omega_2)$ is Weierstrass's $\wp$-function with primitive half-periods ω_1 and ω_2 and has the expansion

$$\wp(z|2\omega_1, 2\omega_2) = \frac{1}{z^2} + \sum_{\omega \ne 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

(cf. [Er], [WW]), where the sum ranges over all $\omega = 2m_1\omega_1 + 2m_2\omega_2$ except 0 (m_1, $m_2 \in \mathbb{Z}$). We allow ω_1 or/and ω_2 to be infinity and

$$\wp(z|\sqrt{-1}\pi, \infty) = \sinh^{-2} z + \frac{1}{3} \quad \text{when} \quad g_2 = \frac{4}{3} \quad \text{and} \quad g_3 = -\frac{8}{27}, \tag{2.3}$$

$$\wp(z|\infty, \infty) = z^{-2} \quad \text{when} \quad g_2 = g_3 = 0. \tag{2.4}$$

Remark. When $u(t) = A_1 \sinh^{-2} t$, the commuting differential operators correspond to J. Sekiguchi's operators.

Theorem 2.4. *Define*

$$\Delta_k = \sum_{0 \le \ell \le \frac{k}{2}} \sum_{g \in W} \frac{1}{\#W(\ell, k-2\ell)} g(L_{\ell, k-2\ell}) \tag{2.5}$$

by putting

$$L_{i,j} = u(x_1 - x_2)u(x_3 - x_4) \cdots u(x_{2i-1} - x_{2i}) \partial_{2i+1} \partial_{2i+2} \cdots \partial_{2i+j},$$

where $W(i,j) = \{g \in W;\ g(L_{i,j}) = L_{i,j}\}$.

Then for any complex numbers A_0, A_1 and any $\wp$-function, $\Delta_1, \ldots, \Delta_n$ are commutative if u is defined by (2.2).

2. Type B_n and D_n

In this section we assume that the Weyl group W is of type B_n with $n \geq 2$ or D_n with $n \geq 4$ (D_3 is isomorphic to A_3). Our problem is to find the W-invariant differential operators

$$P_1 = \partial_1^2 + \cdots + \partial_n^2 + R(x),$$
$$P_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \partial_{i_1}^2 \cdots \partial_{i_k}^2 + R_k(x, \partial) \quad \text{for } 2 \leq k \leq n$$

such that

$$[P_i, P_j] = 0 \quad \text{for } 1 \leq i < j \leq n,$$
$$\operatorname{ord} R_k(x, \partial) < 2k \quad \text{for } 2 \leq k \leq n.$$

When the root system is of type D_n, the operator P_n in the above is replaced by

$$P_n' = \partial_1 \cdots \partial_n + R_n'(x, \partial) \quad \text{with } \operatorname{ord} R_n'(x, \partial) < n.$$

The W-invariance is equivalent to the A_{n-1}-invariance in §1 with the invariance by the coordinate transformation $x_1 \mapsto -x_1$ (resp. $(x_1, x_2) \mapsto (-x_1, -x_2)$) when W is of type B_n (resp. D_n).

Similarly as in the previous section we have

Theorem 3.1. *There exist even holomorphic functions $u(t)$ and $v(t)$ defined for $0 < |t| << 1$ such that*

$$R(x) = \sum_{1 \leq i < j \leq n} \big(u(x_i - x_j) + u(x_i + x_j)\big) + \sum_{1 \leq k \leq n} v(x_k). \tag{3.1}$$

Here $v = 0$ if W is of type D_n.

Theorem 3.2. *The commuting algebra $\mathbb{C}[P_1, \ldots, P_n]$ is uniquely determined by $(u(t), v(t))$ if $\operatorname{ord} R_2(x, \partial) < 3$.*

Remark. Put $Q_i = \partial_i^2 + v(x_i)$ for an even function $v(t)$. Then we easily construct the commuting operators $P_1, \ldots, P_n$ by polynomials of Q_i. We call this a trivial case and this corresponds to the condition $u' = 0$ in Theorem 3.1.

Theorem 3.3. *If W is not of type B_2, then we have*

$$u(t) = A_1 \wp(t | 2\omega_1, 2\omega_2) + A_0,$$
$$v(t) = \frac{C_4 \wp(t)^4 + C_3 \wp(t)^3 + C_2 \wp(t)^2 + C_1 \wp(t) + C_0}{\wp'(t)^2} \tag{3.2}$$

or

$$u(t) = A_1 t^2 + A_2 t^{-2} + A_0,$$
$$v(t) = C_1 t^2 + C_2 t^{-2} + C_0 \tag{3.3}$$

except for the trivial case. Here $C_0 = \cdots = C_4 = 0$ *in the above when* W *is of type* D_n.

The above result corresponds to the necessary and sufficient condition for the existence of P_2 with $[P_1, P_2] = 0$.

If the periods $2\omega_1$ and $2\omega_2$ of $\wp(t)$ are generic, $v(t)$ in (3.2) can be written as

$$v(t) = \sum_{1 \le i \le 3} C_i' \wp(t + \omega_i) + C_4' \wp(t) + C_0' \tag{3.4}$$

with $\omega_3 = -(\omega_1 + \omega_2)$. We remark that $v(t) = C_1'' \wp(t) + C_2'' \wp(2t) + C_0''$ is a special case of (3.4), which corresponds to a non-reduced root system of type BC_n.

In general, we have not yet constructed the commuting differential operators for the potentials defined in Theorem 3.3. If the elliptic functions are degenerated to trigonometric functions (cf. (2.3)) and moreover if

$$\begin{aligned} u(t) &= A_1 \sinh^{-2} \lambda t + A_0, \\ v(t) &= C_1 \sinh^{-2} \lambda t + C_2 \sinh^{-2} 2\lambda t + C_0, \end{aligned}$$

then the potential function coincides with the one studied by Heckman-Opdam and therefore the existence of the commuting differential operators is known.

Proposition 3.4. *Suppose* W *is of type* B_2. *Then we can explicitly construct the commuting differential operators* P_1 *and* P_2 *for* (u, v) *given by* (3.2) *or* (3.3) *but Theorem 3.3 is not true. In fact the functions*

$$\begin{aligned} u(t) &= A_1 \sinh^{-2} \lambda t + A_2 \cosh 2\lambda t + A_0, \\ v(t) &= C_1 \sinh^{-2} \lambda t + C_2 \sinh^{-2} 2\lambda t + C_0 \end{aligned} \tag{3.5}$$

allows the commuting differential operators.

The complete integrability is equivalent to the existence of a symmetric function $T(x, y)$ *of* (x, y) *which satisfies*

$$\frac{\partial}{\partial y} T(x, y) = \frac{1}{2} v'(x)\big(u(x + y) - u(x - y)\big) + v(x)\big(u'(x + y) - u'(x - y)\big). \tag{3.6}$$

If there exist a pair of even functions $(u(t), v(t))$ *and a symmetric function* $T(x, y)$ *satisfying* (3.6), *the following differential operators are commutative:*

$$\begin{aligned} P_1 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + u(x + y) + u(x - y) + v(x) + v(y), \\ P_2 &= \left(\frac{\partial^2}{\partial x \partial y} + \frac{u(x + y) - u(x - y)}{2} \right)^2 + v(y) \frac{\partial^2}{\partial x^2} + v(x) \frac{\partial^2}{\partial y^2} + v(x) v(y) \\ &\quad + T(x, y). \end{aligned} \tag{3.7}$$

4. Examples

First consider the function $v(t)$ given by (3.3) and consider the ordinary differential equation

$$\frac{d^2y}{dt^2} + v(t)y = 0. \tag{4.1}$$

Note that $[\wp']^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$ with some complex numbers g_2, g_3, e_1, e_2 and e_3 and therefore

$$\frac{\wp''}{[\wp']^2} = \frac{1}{2}\Big(\frac{1}{\wp - e_1} + \frac{1}{\wp - e_2} + \frac{1}{\wp - e_3}\Big).$$

Putting $x = \wp(t)$, we have $\frac{d}{dt} = \wp'(t)\frac{d}{dx}$ and

$$\frac{d^2}{dt^2} = [\wp']^2\Big\{\frac{d^2}{dx^2} + \frac{1}{2}\Big(\frac{1}{\wp - e_1} + \frac{1}{\wp - e_2} + \frac{1}{\wp - e_3}\Big)\Big\}$$

and hence (4.1) is transformed into

$$\frac{d^2y}{dx^2} + \frac{1}{2}\Big(\frac{1}{x - e_1} + \frac{1}{x - e_2} + \frac{1}{x - e_3}\Big)\frac{dy}{dx} + \frac{C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0}{16(x - e_1)^2(x - e_2)^2(x - e_3)^2}y = 0. \tag{4.2}$$

Suppose $e_1 \neq e_2 \neq e_3 \neq e_1$. Then (4.2) can be written as

$$\begin{aligned}&\frac{d^2y}{dx^2} + \frac{1}{2}\Big(\frac{1}{x - e_1} + \frac{1}{x - e_2} + \frac{1}{x - e_3}\Big)\frac{dy}{dx}\\ &\quad + \Big(\frac{A_1}{(x - e_1)^2} + \frac{A_2}{(x - e_2)^2} + \frac{A_3}{(x - e_3)^2} + \frac{B_1}{x - e_1} + \frac{B_2}{x - e_2} + \frac{B_3}{x - e_3}\Big)y = 0\end{aligned} \tag{4.3}$$

with some complex numbers A_1, A_2, A_3, B_1, B_2 and B_3 satisfying

$$B_1 + B_2 + B_3 = 0. \tag{4.4}$$

Equation (4.3) is a Fuchsian equation on $\mathbb{P}^1(\mathbb{C})$ which has the four regular singular points e_1, e_2, e_3 and ∞. The indicial equations for the singular points are

$$\begin{aligned}&\rho_j^2 - \frac{1}{2}\rho_j + A_j = 0 \quad \text{at } x = e_j \quad \text{for } j = 1,\ 2 \text{ and } 3,\\ &\rho_\infty^2 - \frac{1}{2}\rho_\infty + \sum_{j=1}^{3}(A_j + e_jB_j) = 0 \quad \text{at } x = \infty.\end{aligned} \tag{4.4}$$

By the transformation $y \mapsto (x - e_1)^{\lambda_1}(x - e_2)^{\lambda_2}(x - e_3)^{\lambda_3}y$ with complex numbers λ_1, λ_2 and λ_3, the equation is transformed into Huen's equation (cf. [WW]) and moreover we obtain any Fuchsian equation on $\mathbb{P}^1(\mathbb{C})$ of order 2 which has the four regular singular points.

On the other hand, if

$$v(t) = C_1 \sinh^{-2} t + C_2 \sinh^{-2} 2t + C_0 \tag{4.5}$$

or

$$v(t) = A_1 \cosh 2t + A_2 + A_0 \tag{4.6}$$

or

$$v(t) = A_1 t^2 + A_2 t^{-2} + A_0 \tag{4.7}$$

(cf. Theorem 3.3 and Proposition 3.4), (4.1) is isomorphic to the Gauss hypergeometric equation or the modified Mathieu equation or the equation of the paraboloid of revolution which is equivalent to the equation of the Whittaker functions, respectively.

If we put $v = u$ for the function u in §1, Theorem 2.3 says $v = C_1\wp + C_0$ and the corresponding equation (4.1) is the Weierstrassian form of Lamé's equation, which corresponds to $A_1 = A_2 = A_3 = 0$ in (4.3). In particular if $v(t) = C_1 \sinh^{-2} t + C_0$ or $v(t) = C_1 t^{-2} + C_0$, the equation reduces to the Legendre equation or the Bessel equation, respectively.

Thus the system $P_1\phi = \cdots = P_n\phi = 0$ with our commuting differential operators $P_1, \ldots, P_n$ is a generalization of these hypergeometric equations to several variables.

We shall give here some examples of type B_2. Let (s, t) be the natural coordinate system of $\mathbb{R}^2$.

When

$$\big(u(t), v(t)\big) = \big(\alpha t^{-2} + \beta t^2, \gamma t^{-2} + \delta t^2\big), \tag{4.8}$$

we have

$$\begin{aligned} P_1 &= \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2} + 2\alpha \frac{s^2 + t^2}{(s^2 - t^2)^2} + (2\beta + \delta)(s^2 + t^2) + \gamma(s^{-2} + t^{-2}), \\ P_2 &= \left[\frac{\partial^2}{\partial s \partial t} - 2\alpha \frac{st}{(s^2 - t^2)^2} + 2\beta st\right]^2 + (\gamma t^{-2} + \delta t^2)\frac{\partial^2}{\partial s^2} + (\gamma s^{-2} + \delta s^2)\frac{\partial^2}{\partial t^2} \\ &\quad + (\gamma s^{-2} + \delta s^2)(\gamma t^{-2} + \delta t^2) + \frac{4\alpha\delta s^2 t^2 + 4\alpha\gamma}{(s^2 - t^2)^2} + 4\beta\delta s^2 t^2. \end{aligned} \tag{4.9}$$

If

$$\big(u(t),\ v(t)\big) = \big(\alpha \sinh^{-2} t + \beta \cosh 2t,\ \gamma \sinh^{-2} t + \delta \sinh^{-2} 2t\big), \tag{4.10}$$

we have

(4.11)

$$
\begin{aligned}
P_1 &= 16x(1+x)\frac{\partial^2}{\partial x^2} + 8(1+2x)\frac{\partial}{\partial x} + 16y(1+y)\frac{\partial^2}{\partial y^2} + 8(1+2y)\frac{\partial}{\partial y} \\
&\quad + 2\alpha\frac{x+y+2xy}{(x-y)^2} + 2\beta(1+2x)(1+2y) \\
&\quad + \gamma\left(\frac{1}{x}+\frac{1}{y}\right) + \delta\left(\frac{1}{4x(1+x)}+\frac{1}{4y(1+y)}\right), \\
P_2 &= \left[16\sqrt{x(1+x)y(1+y)}\frac{\partial^2}{\partial x\partial y} + \left(\frac{-2\alpha}{(x-y)^2}+4\beta\right)\sqrt{x(1+x)y(1+y)}\right]^2 \\
&\quad + \left(\frac{\gamma}{y}+\frac{\delta}{4y(1+y)}\right)\frac{\partial^2}{\partial x^2} + \left(\frac{\gamma}{x}+\frac{\delta}{4x(1+x)}\right)\frac{\partial^2}{\partial y^2} \\
&\quad + \left(\frac{\gamma}{x}+\frac{\delta}{4x(1+x)}\right)\left(\frac{\gamma}{y}+\frac{\delta}{4y(1+y)}\right) + \frac{2\alpha\gamma(2+x+y)+\alpha\delta}{(x-y)^2} + 4\beta\gamma(x+y).
\end{aligned}
$$

by putting $x = \sinh^2 s$ and $y = \sinh^2 t$.

If

(4.12)

$$
\begin{aligned}
u(t) &= A\wp(t), \\
v(t) &= \frac{C_4\wp(t)^4 + C_3\wp(t)^3 + C_2\wp(t)^2 + C_1\wp(t) + C_0}{\wp'(t)^2},
\end{aligned}
$$

we obtain

(4.13)

$$
\begin{aligned}
P_1 &= (4x^3 - g_2x - g_3)\frac{\partial^2}{\partial x^2} + (6x^2 - \frac{g_2}{2})\frac{\partial}{\partial x} + (4y^3 - g_2y - g_3)\frac{\partial^2}{\partial y^2} \\
&\quad + (6y^2 - \frac{g_2}{2})\frac{\partial}{\partial y} + \frac{A(6x^2+6y^2-g_2)}{(x-y)^2} - 2Ax - 2Ay \\
&\quad + \frac{C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0}{4x^3 - g_2x - g_3} + \frac{C_4y^4 + C_3y^3 + C_2y^2 + C_1y + C_0}{4y^3 - g_2y - g_3}, \\
P_2 &= \left[\sqrt{(4x^3 - g_2x - g_3)(4y^3 - g_2y - g_3)}\frac{\partial^2}{\partial x\partial y}\right. \\
&\quad \left. + \frac{A\sqrt{(4x^3 - g_2x - g_3)(4y^3 - g_2y - g_3)}}{2(x-y)^2}\right]^2 \\
&\quad + \frac{C_4y^4 + C_3y^3 + C_2y^2 + C_1y + C_0}{4y^3 - g_2y - g_3}\left((4x^3 - g_2x - g_3)\frac{\partial^2}{\partial x^2} + (6x^2 - \frac{g_2}{2})\frac{\partial}{\partial x}\right) \\
&\quad + \frac{C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0}{4x^3 - g_2x - g_3}\left((4y^3 - g_2y - g_3)\frac{\partial^2}{\partial y^2} + (6y^2 - \frac{g_2}{2})\frac{\partial}{\partial y}\right) \\
&\quad + \frac{(C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0)(C_4y^4 + C_3y^3 + C_2y^2 + C_1y + C_0)}{(4x^3 - g_2x - g_3)(4y^3 - g_2y - g_3)} \\
&\quad + \frac{2AC_4x^2y^2 + AC_3xy(x+y) + 2AC_2xy + AC_1(x+y) + 2AC_0}{2(x-y)^2}.
\end{aligned}
$$

by putting $x = \wp(s)$ and $y = \wp(t)$.

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