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The Null Gauge Condition and the One Dimensional Nonlinear Schrödinger Equation with Cubic Nonlinearity

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In the present note we consider the Cauchy problem of the one dimensional nonlinear Schrödinger equation with cubic nonlinearity:

$$i\frac{\partial u}{\partial t}+\frac{1}{2}D^2u=F(u,Du,\bar{u},D\bar{u}),\quad t>0,\quad x\in\mathbf{R}, \tag{1}$$

$$u(0,x)=u_0(x),\quad x\in\mathbf{R}, \tag{2}$$

where $D=\partial/\partial x$, $\bar{u}$ is the complex conjugate of u and $F(u,Du,\bar{u},D\bar{u})$ is a homogeneous polynomial of degree 3 with respect to u, Du, $\bar{u}$ and $D\bar{u}$. In this note we describe the results concerning the global existence of solutions to (1)-(2) for small initial data, which have recently been obtained in [25].

Let n be the spatial dimensions. When $n\ge 5$ and F is quadratic, the global existence of small amplitude solutions to (1)-(2) was proved by Klainerman [15], Klainerman and Ponce [18] and Shatah [21]. In [15], [18] and [21] they use the L^p-L^q estimate and the energy estimate of the linear Schrödinger equation to show their results. Recently, in [8] Hayashi has proved that when $n=3,4$ and F is quadratic, (1)-(2) has the global solutions for any small initial data. In [8] the clever usage of the generators of the Schrödinger group is a new ingredient of the proof, which reminds us of the results by Klainerman [16] concerning the global existence of small amplitude solution for the nonlinear wave equation. In [16] he uses the generators of the Lorentz group to show his results for the nonlinear wave equation.

Before we consider (1)-(2), we shall recall the results for the nonlinear wave equations. This suggests what happens to the nonlinear Schrödinger equation in the case of $n=1$. The $n+1$ dimensional case for the wave equation corresponds to the n dimensional case for the Schrödinger equation, as is well known. When $n=3$, in [17] Klainerman developed the null condition technique to show the global existence of small amplitude solutions for the nonlinear wave equations with quadratic nonlinearity satisfying a certain algebraic condition, which is called the null condition. Roughly speaking, the null condition is a sufficient condition assuring that the singularity of the solution for the wave equation cancels in the nonlinear terms. When $n=2$, in [7] Godin proves the results analogous to the case of $n=3$ for the nonlinear wave equation with cubic nonlinearity by using the null condition technique (see also Katayama [14]). These results suggest that when $n=1$ and F is cubic, we need consider the new condition assuring the cancellation of singularity in the nonlinear terms for the Schrödinger equation. In the present paper, when $n=1$, we consider a sufficient condition of cubic nonlinearity leading to the global existence of small amplitude solution for (1)-(2). This condition will be called the null gauge condition, because it is closely related to the gauge invariance.

The condition for the nonlinear Klein-Gordon equation corresponding to the null condition for the nonlinear wave equation is studied by Georgiev and Popivanov [6] and Kosecki [19]. Such a condition for the nonlinear Klein-Gordon equation is analogous to the null condition for the nonlinear wave equation (see also Simon and Taflin [22], where they study the global existence and asymptotic behavior of solution for the two dimensional Klein-Gordon equation with quadratic nonlinearity from a different point of view). But the null gauge condition in this paper is different from the both conditions for the nonlinear wave and Klein-Gordon equations.

We first define the null gauge condition for the general space dimensions n as follows.

Definition 1. Let $u, v \in C^1(\mathbf{R}^n)$.

(i) Assume that $f_j(u, v, \bar{u}, \bar{v})$, $1 \leq j \leq n$ are homogeneous polynomials of degree 2 with respect to u, v, $\bar{u}$ and $\bar{v}$ such that

$$f_j(u, v, \bar{u}, \bar{v}) = f_j(e^{i\theta}u, e^{i\theta}v, \overline{e^{i\theta}u}, \overline{e^{i\theta}v}), \quad \theta \in \mathbf{R}, \quad 1 \leq j \leq n. \tag{3}$$

Let a_j, $1 \leq j \leq n$ be the constants in $\mathbf{C}$ such that $\sum_{j=1}^n |a_j|^2 \neq 0$. We shall say that $F(u, \nabla u, v, \nabla v, \bar{u}, \nabla \bar{u}, \bar{v}, \nabla \bar{v})$ satisfies the null gauge condition of order 2, if

$$F(u, \nabla u, v, \nabla v, \bar{u}, \nabla \bar{u}, \bar{v}, \nabla \bar{v}) = \sum_{j=1}^n a_j \frac{\partial}{\partial x_j}[f_j(u, v, \bar{u}, \bar{v})].$$

(ii) Let $f_j(u, v, \bar{u}, \bar{v})$ and a_j, $1 \leq j \leq n$ be defined as in part (i), and let $g_j(u, \nabla u, v, \nabla v, \bar{u}, \nabla \bar{u}, \bar{v}, \nabla \bar{v})$, $1 \leq j \leq n$ be homogeneous polynomials of degree 1 with respect to u, ∇u, v, ∇v, $\bar{u}$, $\nabla \bar{u}$, $\bar{v}$ and $\nabla \bar{v}$. We shall say that F satisfies the null gauge condition of order 3, if

$$\begin{aligned} &F(u, \nabla u, v, \nabla v, \bar{u}, \nabla \bar{u}, \bar{v}, \nabla \bar{v}) \\ &\quad = \sum_{j=1}^n a_j \Big[\frac{\partial}{\partial x_j}\{f_j(u, v, \bar{u}, \bar{v})\}\Big] g_j(u, \nabla u, v, \nabla v, \bar{u}, \nabla \bar{u}, \bar{v}, \nabla \bar{v}). \end{aligned}$$

We state two typical examples of the null gauge condition.

Example (i) We put

$$\begin{aligned} &F(u, \nabla u, v, \nabla v, \bar{u}, \nabla \bar{u}, \bar{v}, \nabla \bar{v}) \\ &\quad = \sum_{j=1}^n a_j \frac{\partial}{\partial x_j}(u\bar{v}) + \sum_{j=1}^n b_j \frac{\partial}{\partial x_j}|u|^2, \end{aligned}$$

where a_j and b_j, $1 \leq j \leq n$ are the constants in $\mathbf{C}$. Then, F satisfies the null gauge condition of order 2.

(ii) We put

$$F(u,\nabla u,v,\nabla v,\bar{u},\nabla\bar{u},\bar{v},\nabla\bar{v})=\sum_{j,k=1}^{n} a_{jk}\Big[\frac{\partial}{\partial x_j}(u\bar{v})\Big]\frac{\partial}{\partial x_k}u,$$

where a_{jk}, $1\le j,k\le n$ are the constants in $\mathbf{C}$. Then, F satisfies the null gauge condition of order 3.

Since the Schrödinger equation is not necessarily stable under the perturbation of lower order unlike the wave equation, we need impose the additional restriction on nonlinearity including the derivative in x. We impose the following assumption on the nonlinear function $F(w,p,z,q)\in C^{\omega}(\mathbf{C}\times\mathbf{C}^n\times\mathbf{C}\times\mathbf{C}^n)$ with $F(0,0,0,0)=0$ for the general space dimensions n:

(E) $\dfrac{\partial}{\partial p_j}F(u,\nabla u,\bar{u},\nabla\bar{u})$ *is a pure imaginary valued function on* $\mathbf{R}^n$ *for* $u\in C^1(\mathbf{R}^n)$.

Remark 1. For convenience, we slightly change the definition of pure imaginary number in this paper. We regard *zero* as pure imaginary throughout this paper. Therefore, (E) allows $\frac{\partial}{\partial p_j}F(u,\nabla u,\bar{u},\nabla\bar{u})$ to take *zero*.

This restriction (E) assures that the linearized Schrödinger equation has the L^2 energy inequality. In other words, (E) implies that (1)-(2) is time locally solvable in the L^2 sense.

The null gauge condition and (E) strongly restrict the form of the admissible nonlinearity. In fact, we have the following proposition.

PROPOSITION 1. *(i) Let n be arbitrary space dimensions. There does not exist $F(u,\nabla u,\bar{u},\nabla\bar{u})$ satisfying both (E) and the null gauge condition of order 2 with $u=v$ in difinition 1(i).*

(ii) Let $n=1$. Assume that $F(u,Du,\bar{u},D\bar{u})$ satisfies both (E) and the null gauge condition of order 3 with $u=v$ in definition 1(ii). Then,

$$F(u,Du,\bar{u},D\bar{u})=i\lambda(D|u|^2)u \tag{4}$$

for some $\lambda\in\mathbf{R}$ with $\lambda\neq 0$.

Remark 2. (i) We are interested in the quadratic nonlinearity for $n=2$ and the cubic nonlinearity for $n=1$. Unfortunately, Porposition 1(i) shows that no quadratic nonlinearity satisfies both (E) and the null gauge condition of order 2 for the case of the decoupled nonlinear Schrödinger equation. However, the null gauge condition of order 2 may be helpful in studying the coupled system of the Schrödinger equations and the wave equations with quadratic nonlinearity such as the Maxwell-Schrödinger equations and the Zakharov equations. Therefore, we formulate the null gauge condition including two functions u and v in definition 1.

(ii) For $n=1$, the null gauge condition of degree 3 and (E) admit only one type of cubic nonlinearity such as (4). However, the nonlinear Schrödinger equation with (4) appears in the nonlinear self-modulation problem of the fluid dynamics (see [23] and [13]).

(iii) The restriction (E) is not a necessary condition but a sufficient condition for the time local solvability in the L^2 sense. In fact, when $n = 1$, we can relax (E) for the local existence of solution to (1)-(2) (see, e.g., Hayashi and Ozawa [12] and Chihara [2]).

Before we state the main theorem in this note, we give several notations. We put $J = x + itD$. For two nonnegative integers m and s, $H^{m,s}$ denotes the weighted Sobolev space defined by

$$H^{m,s} = \{v \in L^2(\mathbf{R}); \|v\|_{H^{m,s}} < +\infty\}$$

with the norm

$$\|v\|_{H^{m,s}} = \|(1 + |x|^2)^{s/2}(1 - D^2)^{m/2}v\|_{L^2}.$$

Let L^p and H^m denote the standard L^p space and the L^2 Sobolev space on $\mathbf{R}$, respectively. Let $U(t) = e^{\frac{i}{2}tD^2}$.

Now we state the main result in this note.

THEOREM 2. *Assume that* $u_0 \in \cap_{j=0}^{2} H^{2-j,j}$. *Then, there exists a* $\delta > 0$ *such that if*

$$\sum_{j=0}^{2} \|u_0\|_{H^{2-j,j}} \leq \delta, \tag{5}$$

then (1.1)-(1.2) with (1.4) has the unique global solution $u(t)$ *satisfying*

$$u(t) \in [\bigcap_{j=0}^{2} C([0,\infty); H^{2-j,j})] \bigcap C^1([0,\infty); L^2), \tag{6}$$

$$\sum_{j+k\leq 2} \sup_{t\geq 0} \|D^j J^k u(t)\|_{L^2} < \infty, \tag{7}$$

$$\sum_{j=0}^{1} \|D^j u(t)\|_{L^\infty} = O(t^{-1/2}) \quad (t \to \infty), \tag{8}$$

where δ *depends only on the coupling constant* λ *in (4). In addition, the above solution* $u(t)$ *of (1)-(2) with (4) has a free profile* $u_{+0} \in H^1$ *such that*

$$\|U(t)u_{+0} - u(t)\|_{H^1} \to 0 \quad (t \to \infty). \tag{9}$$

Remark 3. We know the following two equations similar to (1) with (4):

$$i\frac{\partial u}{\partial t} + \frac{1}{2}D^2 u = \lambda|u|^2 u, \quad t > 0, \quad x \in \mathbf{R}, \tag{10}$$

$$i\frac{\partial u}{\partial t} + \frac{1}{2}D^2 u = i\lambda D(|u|^2 u), \quad t > 0, \quad x \in \mathbf{R}, \tag{11}$$

where $\lambda \in \mathbf{R}$, $\lambda \neq 0$. It is quite interesting to compare the asymptotic behavior in large time of the solution of (1) and (4) with that of the solution of (10) or (11). It is already known that the nontrivial solutions of (10) and (11) have no free profiles in the sense of (9) and that the distortion of the phase of the solutions to (10) and (11) remains as $t \to \infty$ (see [20] for (10) and [11] for (11)). This contrast shows what role the null gauge condition plays in (1).

Remark 4. Hayashi pointed out to the author that equation (1) with (4) could be transformed into the quintic nonlinear Schrödinger equation by the gauge transformation:

$$v(t,x) \equiv \exp(-i\lambda \int_{-\infty}^{x} |u(t,y)|^2 \, dy) \, u(t,x).$$

In [26], it is proved that in Theorem 2 the restriction (E) can be replaced by the gauge invariance of the equation (1), which is an extension of Theorem 2. The author does not know whether all the equations (1) with gauge covariant cubic nonlinearity can be transformed into new equations with quintic nonlinearity.

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