数理解析研究所講究録
第 862 巻 1994 年 35-45

Asymptotic Behavior of the Solutions to a One-Dimensional Motion of Compressible Viscous Fluids II

Shigenori Yanagi 愛媛大・理 柳 重則

Department of Mathematics
Ehime University
Matsuyama, 790 Japan

Abstract We study the one-dimensional motion of the viscous gas represented by the system $v_t - u_x = 0, u_t + p(v)_x = \mu(u_x/v)_x + f\left(\int_0^x v dx, t\right)$, with the initial and the boundary conditions $(v(x,0), u(x,0)) = (v_0(x), u_0(x))$, $u(0,t) = u(X,t) = 0$. We are concerned about the external forces, namely the function f, which do not become small for large time t. The main purpose is to show how the solution to this problem behaves around the stationary one, and the proof is based on an elementary L^2-energy method.

1 Introduction

In this paper we study the asymptotic behavior of the solutions to a one-dimensional motion of the viscous gas on a finite interval. In Lagrangian mass coordinate, such a motion is described by the following system of equations

(1.1) $$v_t - u_x = 0,$$

(1.2) $$u_t + p(v)_x = \mu\left(\frac{u_x}{v}\right)_x + f\left(\int_0^x v dx, t\right),$$

where v, u, p, μ, and f are the specific volume, the velocity, the pressure, the viscosity coefficient, and the external force of the fluid, respectively. We consider these equations in a fixed domain Q_∞ defined by

(1.3) $$Q_\infty = \{\ (x,t) \mid 0 < x < X,\ t > 0\},$$

together with the following initial and the boundary conditions

(1.4) $$v(x,0) = v_0(x),\ \ u(x,0) = u_0(x) \quad \text{on} \ \ 0 < x < X,$$

(1.5) $$u(0,t) = u(X,t) = 0 \quad \text{on} \ \ t > 0.$$

This and related problems have been investigated by a number of authors including Kanel' [5],Itaya [3, 4],Kazhikhov [6], Kazhikhov & Shelukhin [9],Kazhikhov & Nikolaev [7, 8], and so on. For their results and the historical progress, we could refer to the paper of Solonnikov & Kazhikhov [12].

Now we proceed to review this problem in terms of the presence of external forces. Matsumura & Nishida [11] proved the global existence of the solution for any external forces with its derivatives and itself being bounded, assuming that the gas is isothermal, and obtained the following estimate

$$C_0^{-1} \le v(x,t) \le C_0 \quad \text{for } (x,t) \in Q_\infty, \tag{1.6}$$

where C_0 is a positive constant. Recently, Matsumura [10] improved their results, showing that the solution is exponentially stable if the external force depends only on $\xi = \int_0^x v dx$ and its derivative with respect to ξ is sufficiently small. For a general barotropic gas, Tani obtained in his lecture note [13] the exponential stability of the solution if $f(\xi,t)$ belongs to $L^1(0,\infty;L^\infty(I)) \cap L^2(I \times (0,\infty))$, where $I = [0, \int_0^X v_0 dx]$. We shall also mention about the papers of Beirão da Veiga. In [2], he proved the global existence of the solution if some norm of the initial date is bounded by some constant which is determined by the L^∞-norm of f. We notice that his conclusion is not a result for small date, because the constant mentioned above tends to infinity as the L^∞-norm of f tends to 0. In [1], he had also obtained, in his words, the complete characterization of time independent external forces for which corresponding stationary solutions are known to exist (see also [2]). Finally, we shall show Zlotnik's interesting results. In [16], he proved that if the stationary state of the external force is a decreasing function of ξ, then the solution is exponentially stable.

Our interest in the present paper is to investigate the asymptotic behavior of the solution with external forces depending on time t and not becoming small for large time. We will consider two cases, namely we will investigate an ideal gas in section 2, and a general barotropic gas in section 3. In what follows, we assume that that the viscosity coefficient is a positive constant, and that the external force $f = f(\xi,t)$, $\xi = \int_0^x v dx$ has a limit function $\hat{f}(\xi)$ in $L^\infty(I)$ satisfying

$$f_0(\xi,t) \equiv f(\xi,t) - \hat{f}(\xi) \in L^2(0,\infty;L^\infty(I)), \tag{1.7}$$

where $I = [0, \int_0^X v_0 dx]$. To obtain the *strong solution* (see [2], for example), we impose the following assumptions on the initial data and the external force

$$(v_0, u_0) \in H^1(0,X) \times H_0^1(0,X), \quad \inf_x v_0(x) > 0, \tag{1.8}$$

$$f, \; f_\xi, \; \text{and } f_t \in L^\infty(I \times (0,\infty)), \tag{1.9}$$

where H^k and H_0^k $(k \ge 0)$ are the usual Sobolev's spaces with the norm $\|\cdot\|_k$, and we use the notation $\|\cdot\|$ instead of $\|\cdot\|_0$.

2 In case of $p = \dfrac{a}{v}$

2.1 The Stationary Problem and the Theorem

In this section, we asuume that the gas is ideal, i.e.

$$p(v) = \frac{a}{v} \quad (a, \text{ positive constant}). \tag{2.1}$$

Then the equation (1.2) is reduced to

$$u_t + (\frac{a}{v})_x = \mu \left(\frac{u_x}{v}\right)_x + f\left(\int_0^x vdx, t\right). \tag{2.2}$$

For the global existence of the solution to our system, we have already known the following theorem [11]

Theorem 2.1 *(Matsumura & Nishida) Assume (1.8) and (1.9). Then the initial and boundary value problem (1.1),(1.4),(1.5),(2.2) has an unique global solution in* $C^0([0,\infty); H^1 \times H_0^1)$ *satisfying (1.6) and the following estimate*

$$\sup_{t\geq 0} \| (v,u)(t) \|_1 \leq C(\| (v_0,u_0) \|_1, \inf_x v_0, |f|_\infty). \tag{2.3}$$

In order to investigate the asymptotic behavior of the solution, it is necessary to consider the stationary problem. Let $(\eta(x), 0)$ be the stationary solution to (1.1),(1.4),(1.5) and (2.2), then the function $\eta(x)$ must satisfy the following system of equations

$$\left(\frac{a}{\eta}\right)_x = \hat{f}\left(\int_0^x \eta dx\right), \tag{2.4}$$

$$\int_0^X \eta(x)dx = \int_0^X v_0(x)dx \quad (\equiv Y). \tag{2.5}$$

We can easily see that this stationary problem has an unique solution in the following way. Let $w(x)$ be defined by $w(x) = \int_0^x \eta dx$. Then (2.4) and (2.5) are reduced to

$$\left(\frac{a}{w_x}\right)_x = \hat{f}(w), \tag{2.6}$$

$$w(0) = 0, \quad w(X) = Y. \tag{2.7}$$

We rewrite (2.6) as follows

$$-a\frac{w_{xx}}{w_x} = F(w)_x, \tag{2.8}$$

where $F(w)$ is defined by $F(w) = \int_0^w \hat{f}(\xi)d\xi$. Integration of (2.8) with respect to x implies

$$w_x = Ae^{-\frac{1}{a}F(w)}, \tag{2.9}$$

here A is a constant. Since $F(w)$ is a Lipschitz continuous function, the initial value problem (2.9) with $w(0) = 0$ in (2.7) has an unique solution for arbitrary fixed constant A. We now proceed to show that there is an unique constant A for which the above solution satisfies the relation $w(X) = Y$ in (2.7). As the proof of the existence is trivial, we shall only prove the uniqueness. We note that $A > 0$ because of $Y > 0$. Let A and B satisfy $A > B(> 0)$, and w_A, w_B be the corresponding unique solutions to (2.9) with $w(0) = 0$. It is enough to show that $w_A(x) > w_B(x)$ for $0 < x \leq X$. We shall prove it by reductio ad absurdum. We assume that there exists a point $x_0 \in (0, X]$, such that $w_A(x_0) = w_B(x_0)$ and $w_A(x) > w_B(x)$ for $0 < x < x_0$. Then we must have $w_{Ax}(x_0) \leq w_{Bx}(x_0)$. On the other hand, from (2.9), we have $w_{Ax}(x_0) > w_{Bx}(x_0)$. This is a contradiction.

Then our first main theorem is

Theorem 2.2 *Assume (1.7) - (1.9). Let (v,u) be the unique global solution to (1.1),(1.4),(1.5), (2.2), and η be the stationary solution mentioned above. Then there exist constants $\epsilon_0 > 0$, $\delta > 0$ and $C > 0$ which depend only on the given data such that if $|f_\xi|_\infty \leq \epsilon_0$ then the following estimate is satisfied for all $t \geq 0$*

$$\| (v-\eta)(t) \|_1^2 + \| u(t) \|_1^2 \leq Ce^{-\delta t}\left(1 + \int_0^t e^{\delta s}|f_0(s)|_\infty^2 ds\right). \tag{2.10}$$

The proof of this theorem is done in section 2.3. In section 2.2, we will show some energy estimates used in section 2.3.

2.2 Energy Estimates

In what follows, we shall denote the letter C by an universal constant which depends only on the given data. We first prove the following lemma.

Lemma 2.1 *Let (v,u) be the unique solution of (1.1),(1.4),(1.5),(2.2), and η be the unique solution of (2.4),(2.5). Then the following estimate is valid for all $t \geq 0$*

$$\frac{d}{dt}\int_0^X \{\frac{1}{2}u^2 + P(v,\eta)\}dx + \frac{\mu}{2}\int_0^X \frac{u_x^2}{v}dx \leq C\left(|f_\xi|_\infty \int_0^X vQ_x^2 dx + |f_0(t)|_\infty^2\right), \tag{2.11}$$

where P and Q are defined by $P(v,\eta) = a(\frac{v}{\eta} + \log\frac{\eta}{v} - 1) \geq 0$ and $Q = \frac{a}{v} - \frac{a}{\eta}$, respectively, and where $|f_\xi|_\infty$ denotes the $L^\infty(I \times (0,\infty))$-norm of f_ξ, on the other hand, $|f_0(t)|_\infty$ denotes the $L^\infty(I)$-norm of f_0.

Proof. We rewrite the equation (2.2) in the form

$$\begin{aligned} u_t + Q_x &= \mu\left(\frac{u_x}{v}\right)_x + f\left(\int_0^x vdx, t\right) - \hat{f}\left(\int_0^x \eta dx\right) \\ &= \mu\left(\frac{u_x}{v}\right)_x + f\left(\int_0^x vdx, t\right) - f\left(\int_0^x \eta dx, t\right) + f\left(\int_0^x \eta dx, t\right) - \hat{f}\left(\int_0^x \eta dx\right) \\ &= \mu\left(\frac{u_x}{v}\right)_x + f_\xi(\cdot, t)\int_0^x (v-\eta)dx + f_0\left(\int_0^x \eta dx, t\right), \end{aligned} \tag{2.12}$$

where we have used the relation (2.4). We multiply (1.1) by $-Q$, (2.12) by u and add the results. Integration of this equation over $[0,X]$ yields

$$\frac{d}{dt}\int_0^X \{\frac{1}{2}u^2 + P(v,\eta)\}dx + \mu\int_0^X \frac{u_x^2}{v}dx = \int_0^X f_\xi u dx \int_0^x (v-\eta)dx' + \int_0^X f_0 u dx. \tag{2.13}$$

Using (1.6) and the relation $\| u \| \leq C \| u_x \|$, each term of the right hand side of (2.13) is estimated as follows

$$\begin{aligned} \left|\int_0^X f_\xi u dx \int_0^x (v-\eta)dx'\right| &\leq |f_\xi|_\infty \int_0^X |u|dx \int_0^x |v-\eta|dx' \\ &\leq \frac{\mu}{4}\int_0^X \frac{u_x^2}{v}dx + C|f_\xi|_\infty \int_0^X vQ^2 dx, \end{aligned} \tag{2.14}$$

$$\left|\int_0^X f_0 u dx\right| \le \frac{\mu}{4}\int_0^X \frac{u_x^2}{v}dx + C|f_0(t)|_\infty^2. \tag{2.15}$$

As discussed in [14],there exists $X_1(t) \in [0, X]$ such that $v(X_1(t), t) = \eta(X_1(t))$, so that Q can be represented by $Q = \int_{X_1(t)}^x Q_x dx$, which gives the relation $\| Q \| \le C \| Q_x \|$. From (2.13)-(2.15) and the above relation, we obtain (2.11). ∎

In the next lemma, we shall estimate Q_x.

Lemma 2.2 *Under the same situation as in Lemma 2.1, the following estimate is satisfied for all* $t \ge 0$

$$\begin{aligned} &\frac{d}{dt}\int_0^X \{\frac{\mu}{2a}(vQ_x)^2 + uvQ_x\}dx + \left(\frac{1}{2} - C|f_\xi|_\infty\right)\int_0^X vQ_x^2 dx \\ &\le C\left(\int_0^X \frac{u_x^2}{v}dx + |f_0(t)|_\infty^2\right). \end{aligned} \tag{2.16}$$

Proof. Owing to the relation $v_t = u_x$, it is easy to see that

$$(vQ_x)_t + \left(\frac{a}{\eta}\right)_x u_x = \left(-\frac{au_x}{v}\right)_x. \tag{2.17}$$

Thus we can rewrite (2.12) in the form

$$u_t + Q_x + \frac{\mu}{a}(vQ_x)_t + \frac{\mu}{a}\hat{f}\left(\int_0^x \eta dx\right)u_x = f_\xi(\cdot, t)\int_0^x (v - \eta)dx + f_0\left(\int_0^x \eta dx, t\right). \tag{2.18}$$

Multiplying (2.18) by vQ_x and integrating it over $[0, X]$ give

$$\begin{aligned} &\frac{\mu}{2a}\frac{d}{dt}\int_0^X (vQ_x)^2 dx + \int_0^X vQ_x^2 dx + \int_0^X u_t vQ_x dx + \frac{\mu}{a}\int_0^X \hat{f}u_x vQ_x dx \\ &= \int_0^X f_\xi vQ_x dx\int_0^x (v - \eta)dx' + \int_0^X f_0 vQ_x dx. \end{aligned} \tag{2.19}$$

The third term of the left hand side of (2.19) is calculated as follows

$$\begin{aligned} &\int_0^X u_t vQ_x dx \\ &= \frac{d}{dt}\int_0^X uvQ_x dx - \int_0^X u(vQ_x)_t dx \\ &= \frac{d}{dt}\int_0^X uvQ_x dx + \int_0^X u\left\{\hat{f}u_x + \left(\frac{au_x}{v}\right)_x\right\}dx \\ &= \frac{d}{dt}\int_0^X uvQ_x dx - \int_0^X \frac{au_x^2}{v}dx + \int_0^X uu_x\hat{f}dx, \end{aligned} \tag{2.20}$$

where we have used (2.17). By using (1.6) and Schwarz's inequality, it follows from (2.19) and (2.20) that

$$\frac{d}{dt}\int_0^X \left\{\frac{\mu}{2a}(vQ_x)^2 + uvQ_x\right\}dx + \int_0^X vQ_x^2 dx \tag{2.21}$$

$$
\begin{aligned}
&= \int_0^X \frac{a u_x^2}{v} dx - \frac{\mu}{a} \int_0^X \hat{f} u_x v Q_x dx - \int_0^X u u_x \hat{f} dx \\
&\quad + \int_0^X f_\xi v Q_x dx \int_0^x (v-\eta) dx' + \int_0^X f_0 v Q_x dx \\
&\le a \int_0^X \frac{u_x^2}{v} dx + \frac{1}{4} \int_0^X v Q_x^2 dx + C \int_0^X \frac{u_x^2}{v} dx \\
&\quad + C |f_\xi|_\infty \int_0^X v Q_x^2 dx + \frac{1}{4} \int_0^X v Q_x^2 dx + C |f_0(t)|_\infty^2 \\
&= \left(\frac{1}{2} + C |f_\xi|_\infty \right) \int_0^X v Q_x^2 dx + C \left(\int_0^X \frac{u_x^2}{v} dx + |f_0(t)|_\infty^2 \right).
\end{aligned}
$$

This completes the proof of Lemma 2.2. ■

We finally estimate u_x.

Lemma 2.3 *Under the same situation as in Lemma 2.1, we have the following estimate for all* $t \ge 0$

$$
\frac{1}{2} \frac{d}{dt} \int_0^X u_x^2 dx + \frac{\mu}{2} \int_0^X \frac{u_{xx}^2}{v} dx \le C \left(\int_0^X \frac{u_x^2}{v} dx + \int_0^X v Q_x^2 dx + |f_0(t)|_\infty^2 \right). \tag{2.22}
$$

Proof. Multiplying (2.12) by $-u_{xx}$ and integrating it over $[0, X]$ yield

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_0^X u_x^2 dx + \mu \int_0^X \frac{u_{xx}^2}{v} dx \\
&= \int_0^X Q_x u_{xx} dx + \mu \int_0^X \frac{v_x u_x u_{xx}}{v^2} dx \\
&\quad - \int_0^X f_\xi u_{xx} dx \int_0^x (v-\eta) dx' - \int_0^X f_0 u_{xx} dx.
\end{aligned} \tag{2.23}
$$

Each term of the right hand side of (2.23) is estimated as follows. First by using Schwarz's inequality,

$$
\left| \int_0^X Q_x u_{xx} dx \right| \le \frac{\mu}{10} \int_0^X \frac{u_{xx}^2}{v} dx + C \int_0^X v Q_x^2 dx. \tag{2.24}
$$

Next,

$$
\mu \left| \int_0^X \frac{v_x u_x u_{xx}}{v^2} dx \right| \le \frac{\mu}{10} \int_0^X \frac{u_{xx}^2}{v} dx + C \int_0^X v_x^2 u_x^2 dx. \tag{2.25}
$$

Because of $u(0,t) = u(X,t) = 0$, there exists $X_2(t) \in [0, X]$ such that $u_x(X_2(t), t) = 0$, so that

$$
u_x^2 = \int_{X_2(t)}^x \frac{\partial}{\partial x} u_x^2 dx \le 2 \int_0^X |u_x u_{xx}| dx \le \epsilon \int_0^X u_{xx}^2 dx + C \int_0^X u_x^2 dx \tag{2.26}
$$

for any small $\epsilon > 0$. Therefore, the last term of the right hand side of (2.25) is estimated as follows

$$
C \int_0^X v_x^2 u_x^2 dx \le \frac{\mu}{10} \int_0^X \frac{u_{xx}^2}{v} dx + C \int_0^X \frac{u_x^2}{v} dx. \tag{2.27}
$$

Here we have used (2.3). Next,

$$\left|\int_0^X f_\xi u_{xx} dx \int_0^x (v-\eta) dx'\right| \le \frac{\mu}{10}\int_0^X \frac{u_{xx}^2}{v} dx + C\int_0^X vQ_x^2 dx. \tag{2.28}$$

Finally,

$$\left|\int_0^X f_0 u_{xx} dx\right| \le \frac{\mu}{10}\int_0^X \frac{u_{xx}^2}{v} dx + C|f_0(t)|_\infty^2. \tag{2.29}$$

From inserting above inequalities (2.24)-(2.29) into (2.23), it immediately follows (2.22). ■

2.3 Proof of Theorem 2.2

We are now in a position to show the Theorem 2.2. Multiplying (2.16) by θ_1,(2.22) by θ_2 and adding the results together with (2.11) imply

$$\begin{aligned} &\frac{d}{dt}E^2(t) + \left(\frac{\mu}{2} - C\theta_1 - C\theta_2\right)\int_0^X \frac{u_x^2}{v} dx \\ &+ \left(\frac{\theta_1}{2} - C(1+\theta_1)|f_\xi|_\infty - C\theta_2\right)\int_0^X vQ_x^2 dx + \frac{\mu\theta_2}{2}\int_0^X \frac{u_{xx}^2}{v} dx \\ &\le C(1+\theta_1+\theta_2)|f_0(t)|_\infty^2, \end{aligned} \tag{2.30}$$

where $E^2(t)$ is defined by

$$E^2(t) = \int_0^X \left\{\frac{1}{2}u^2 + P(v,\eta) + \frac{\mu\theta_1}{2a}(vQ_x)^2 + \theta_1 uvQ_x + \frac{\theta_2}{2}u_x^2\right\} dx. \tag{2.31}$$

Using Schwarz's inequality, we can estimate the term $\theta_1 uvQ_x$ as follows

$$|\theta_1 uvQ_x| \le \frac{\mu\theta_1}{4a}(vQ_x)^2 + \frac{a\theta_1}{\mu}u^2. \tag{2.32}$$

Thus if $|f_\xi|_\infty$ is sufficiently small, we can choose the positive constants θ_1 and θ_2 to be satisfied

$$\frac{\mu}{2} - C\theta_1 - C\theta_2 > 0,\ \frac{\theta_1}{2} - C(1+\theta_1)|f_\xi|_\infty - C\theta_2 > 0,\ \frac{1}{2} - \frac{a\theta_1}{\mu} > 0, \tag{2.33}$$

so that $E^2(t) \ge 0$, and the coefficient of the second and the third term of the left hand side of (2.30) is positive. We note that because of (1.6), P and Q^2 are equivalent. Furthermore, as stated in section 2.2, we have the relation $\| Q \| \le C \| Q_x \|$. Thus it follows from these remarks and (2.30) that there exists a positive constant δ such that

$$\frac{d}{dt}E^2(t) + \delta E^2(t) \le C|f_0(t)|_\infty^2 \tag{2.34}$$

holds for all $t \ge 0$. From which, we obtain

$$E^2(t) \le E^2(0)e^{-\delta t} + Ce^{-\delta t}\int_0^t e^{\delta s}|f_0(s)|_\infty^2 ds. \tag{2.35}$$

It is easy to see that (2.35) implies (2.10), and the proof of theorem 2.2 is completed. ■

3 In case of $p = av^{-\gamma}, \quad \gamma > 1$

3.1 The Stationary Problem and the Theorem

In this section, we consider the general barotropic gas represented by

$$p(v) = av^{-\gamma} \quad (a > 0, \gamma > 1 \ \text{constants}). \tag{3.1}$$

Then the equation (1.2) is reduced to

$$u_t + \left(\frac{a}{v^\gamma}\right)_x = \mu\left(\frac{u_x}{v}\right)_x + f\left(\int_0^x v dx, t\right). \tag{3.2}$$

As mentioned in section 1, we have already known the following global existence theorem [2]

Theorem 3.1 *(H. Beirão da Veiga) Assume (1.8) and (1.9). Then there exists a decreasing function $R(\cdot)$ satisfying $R(0) = \infty$ such that if $\| (v_0, u_0) \|_1 \leq R(|f|_\infty)$ then the initial and boundary value problem (1.1),(1.4),(1.5),(2.2) has an unique global solution in $C^0([0,\infty); H^1 \times H_0^1)$ satisfying (1.6) and (2.3).*

Stationary problem considered in this section is the following

$$\left(\frac{a}{\eta^\gamma}\right)_x = \hat{f}\left(\int_0^x \eta dx\right), \tag{3.3}$$

$$\int_0^X \eta(x) dx = \int_0^X v_0(x) dx \quad (\equiv Y). \tag{3.4}$$

Proceeding the same calculation as in section 2.1, (3.3) and (3.4) are rewritten as

$$\Phi(w_x)_x = F(w)_x, \tag{3.5}$$

$$w(0) = 0, w(X) = Y, \tag{3.6}$$

here $w(x), \Phi(w)$, and $F(w)$ are defined by $w(x) = \int_0^x \eta dx, \Phi(w) = \frac{a\gamma}{\gamma - 1}(w^{1-\gamma} - 1)$, and $F(w) = \int_0^w \hat{f}(\xi) d\xi$.

From (3.5), we have

$$\Phi(w_x) = F(w) + c, \tag{3.7}$$

where c is a constant. Let M and m be defined by $M = \max_{0 \leq w \leq Y} F(w)$ and $m = \min_{0 \leq w \leq Y} F(w)$. Then we must have

$$m + c > -\frac{a\gamma}{\gamma - 1} \quad \left(= \inf_{0 \leq w \leq Y} \Phi(w) \right), \tag{3.8}$$

because we are looking for a solution that satisfies $\inf_{0\le x\le X}\eta(x) > 0$. Now let us fix a constant c that satisfies (3.8). Since $\Phi(w)$ is a decreasing function of w, we can solve (3.7) as

$$w_x = \Phi^{-1}\left(F(w)+c\right). \tag{3.9}$$

It is easy to see that the initial value problem (3.9) with $w(0) = 0$ in (3.6) has an unique solution for arbitrary fixed constant c satisfying (3.8), and we denote this solution by $w_c(x)$. The unique existence of a constant c satisfying $w_c(X) = Y$ is our problem. As the proof of the uniqueness is easily verified by using the comparison theorem, we shall only consider the existence. Integration of (3.9) over $[0, X]$ yields

$$Y = \int_0^X \Phi^{-1}\left(F(w)+c\right)dx. \tag{3.10}$$

Thus the necessary and sufficient condition for the existence is given by

$$\lim_{c\to -m-\frac{a\gamma}{\gamma-1}} \int_0^X \Phi^{-1}\left(F(w)+c\right)dx > Y. \tag{3.11}$$

From which, we obtain one of the sufficient condition as follows

$$\Phi\left(\frac{Y}{X}\right) > M - m - \frac{a\gamma}{\gamma-1}. \tag{3.12}$$

Then our final main theorem is

Theorem 3.2 *Assume the hypotheses in Theorem 3.1 and the existence of the stationary solution. Then there exist constants $\epsilon_0 > 0$, $\delta > 0$ and $C > 0$ which depend only on the given data such that if $|f_\xi|_\infty \le \epsilon_0$ then the following estimate is satisfied for all $t \ge 0$*

$$\| (v-\eta)(t) \|_1^2 + \| u(t) \|_1^2 \le Ce^{-\delta t}\left(1 + \int_0^t e^{\delta s}|f_0(s)|_\infty^2 ds\right). \tag{3.13}$$

The proof of this theorem is similar to that of Theorem 2.2, so we will only show the sketch of proof in the next subsection.

3.2 Sketch of Proof of Theorem 3.2

As in section 2.2, we derive the following three energy estimates.

Lemma 3.1 *Let (v, u) be the unique solution of (1.1),(1.4),(1.5),(3.2), and η be the unique solution of (3.3),(3.4). Then the following estimate is valid for all $t \ge 0$*

$$\frac{d}{dt}\int_0^X \left\{\frac{1}{2}u^2 + P(v,\eta)\right\}dx + \frac{\mu}{2}\int_0^X \frac{u_x^2}{v}dx \le C\left(|f_\xi|_\infty \int_0^X v^\gamma Q_x^2 dx + |f_0(t)|_\infty^2\right), \tag{3.14}$$

where P and Q are defined by $P(v,\eta) = a\left(\frac{1}{\gamma-1}v^{-\gamma+1} + v\eta^{-\gamma} - \frac{\gamma}{\gamma-1}\eta^{-\gamma+1}\right) \ge 0$ *and* $Q = \frac{a}{v^\gamma} - \frac{a}{\eta^\gamma}$*, respectively.*

Lemma 3.2 *Under the same situation as in Lemma 3.1, the following estimate is satisfied for all* $t \geq 0$

$$\frac{d}{dt}\int_0^X \{\frac{\mu}{2a\gamma}(v^\gamma Q_x)^2 + uv^\gamma Q_x\}dx + \left(\frac{1}{2} - C|f_\xi|_\infty\right)\int_0^X v^\gamma Q_x^2 dx \leq C\left(\int_0^X \frac{u_x^2}{v}dx + |f_0(t)|_\infty^2\right). \tag{3.15}$$

Lemma 3.3 *Under the same situation as in Lemma 3.1, we have the following estimate for all* $t \geq 0$

$$\frac{1}{2}\frac{d}{dt}\int_0^X u_x^2 dx + \frac{\mu}{2}\int_0^X \frac{u_{xx}^2}{v}dx \leq C\left(\int_0^X \frac{u_x^2}{v}dx + \int_0^X v^\gamma Q_x^2 dx + |f_0(t)|_\infty^2\right). \tag{3.16}$$

The proof of those lemmas is done by the same procedure as in Lemma 2.1 - Lemma 2.3, and we ommit it only noting that we use the following relation in Lemma 3.2 instead of (2.17).

$$(v^\gamma Q_x)_t + \left(\frac{a}{\eta^\gamma}\right)_x \gamma v^{\gamma-1}u_x = -\gamma a\left(\frac{u_x}{v}\right)_x. \tag{3.17}$$

Now the proof of Theorem 3.2 is easy; with these three inequalities, the same consideration as in section 2.3 leads Theorem 3.2.

References

[1] H. Beirão da Veiga; An L^p-Theory for the n-Dimensional,Statinary, Compressible, Navier-Stokes Equations, and the Incompressible Limit for Compressible Fluids. The Equilibrium Solutions, Comm.Math.Physics, **109**,229-248(1987).

[2] H. Beirão da Veiga; Long Time Behavior for One-Dimensional Motion of a General Barotropic Viscous Fluid, Arch.Rat.Mech.Anal,**108**,141-160 (1989).

[3] N. Itaya; The Existence and Uniqueness of the Solution of the Equations Describing Compressible Viscous Fluid Flow, Proc. Jpn. Acad., **46**,379-382(1970).

[4] N. Itaya; A Survey on the Generalized Burger's Equation with Pressure Model Term, J.Math.Kyoto Univ.,**16**,223-240(1976).

[5] Ya. Kanel'; On a Model System of Equations of One-Dimensional Gas Motion, Diff. Eqns.,**4**,374-380(1968).

[6] A. V. Kazhikhov; Correctness "In the Large" of Initial-Boundary-Value Problem for Model System of Equations of a Viscous Gas, Din.Sploshnoi Sredy,**21**,18-47(1975).

[7] A. V. Kazhikhov and V. B. Nikolaev; On the Correctness of Boundary Value Problems for the Equations of a Viscous Gas with a Non-Monotonic Function of State, Chislennye Metody Mekh.Sploshnoi Sredy,**10**,77-84(1979).

[8] A. V. Kazhikhov and V. B. Nikolaev; On the Theory of the Navier-Stokes Equations of a Viscous Gas with Nonmonotone State Function, Soviet Math. Dokl.,**20**,583-585(1979).

[9] A. V. Kazhikhov and V. V. Shelukhin; Unique Global Solution with Respect to Time of Initial-Boundary Value Problems for One-Dimensional Equations of a Viscous Gas, J.Appl.Math.Mech.,**41**,273-282(1977)

[10] A. Matsumura; Large Time Behavior of the Solutions of a One-Dimensional Barotropic Model of Compressible Viscous Gas (preprint).

[11] A. Matsumura and T. Nishida; Periodic Solutions of a Viscous Gas Equation, Lec.Notes in Num. Appl.Anal,**10**,49-82(1989).

[12] V. A. Solonnikov and A. V. Kazhikhov; Existence Theorems for the Equations of Motion of a Compressible Viscous Fluid, Ann.Rev.Fluid Mech.,**13**,79-95(1981).

[13] A. Tani; A Survey on the One-Dimensional Compressible Isentropic Navier-Stokes Equations in a Field of External Forces (unpublished)

[14] S. Yanagi; Global Existence for One-Dimensional Motion of Non-Isentropic Viscous Fluids, Math.Methods in Appl.Sci.,**16**,609-624(1993).

[15] S. Yanagi; Asymptotic Behavior of the Solutions to a One-Dimensional Motion of Compressible Viscous Fluids (preprint).

[16] A. A. Zlotnik; On Equations for One-Dimensional Motion of a Viscous Barotropic Gas in the Presence of a Body Force, Sibir.Mat.Zh.,**33**,62-79(1993).