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On the convergency to the limit cycle in the dynamical system of Multivibrator

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Abstract

The dynamics of a multivibrator circuit with two slow and two fast degrees of freedom in the singular limit depends on only one parameter. For some values of it, the trajectories of the system take place jumping. This paper, by using a quasi-potential in this system, shows that the trajectories converge to the limit cycle where repeated jumping behaviour is observed.

1. Introduction.

The fast vector field gives 2-dimensional manifold in R^4 at the singular limit ($\varepsilon \to 0$) and the slow vector field on the manifold describes the dynamics. Therefore, we consider the natural projection $\Pi_2 : R^4 \to R^2$, determined by the translated coordinate. Then, it is easy to analyze the system by using approximated FET (field effect transistor) characteristic as a piecewise linear function. This paper proves the convergence of the trajectories to the limit cycle through the function. Judging the convergency, we use a quasi-potential for the system instead of the trajectory itself.

2. The dynamical system of multivibrator.

In this system, we get the following equations([1]):

$$\begin{pmatrix} \dot{v}_3 \\ \dot{v}_6 \end{pmatrix} = -1/(CR_c) \begin{pmatrix} v_3+v_{10} \\ v_6+v_{11} \end{pmatrix} \qquad (\dot{\ } = d/dt) \qquad \equiv F(v_3, v_6, v_{10}, v_{11}), \quad (1)$$

$$\varepsilon \begin{pmatrix} \dot{v}_{10} \\ \dot{v}_{11} \end{pmatrix} = \begin{pmatrix} -(v_3+v_{10})/R_c + f(-v_6-v_{11}) - v_{10}/R_a \\ -(v_6+v_{11})/R_c + f(-v_3-v_{10}) - v_{11}/R_a \end{pmatrix} \equiv G(v_3, v_6, v_{10}, v_{11}), \quad (2)$$

where ε is a parasitic inductance (any $\varepsilon \in R^1{}_+$) and f is defined as follows ($f \in C^0$):

$$f(x) = \begin{cases} 0 & (x \geqq \alpha) \\ (\beta/\alpha)x - \beta & (|x| \leqq \alpha) \\ -2\beta & (x \leqq -\alpha) \end{cases} \quad (3)$$

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Then, we assume that

$$\beta/\alpha > 1/R_a + 1/R_c \equiv 1/R\ . \tag{4}$$

We denote the following notations:

$$\Sigma \equiv \{(v_3, v_6, v_{10}, v_{11});\ G=0\}, \tag{5}$$

$$\Sigma r \equiv \{(v_3, v_6, v_{10}, v_{11}) \in \Sigma;\ \ \partial G/\partial(v_{10}, v_{11}) \text{ is nondegenerate}\},$$

$$(\partial\Sigma = \{(v_3, v_6, v_{10}, v_{11}) \in \Sigma;\ \ |\partial G/\partial(v_{10}, v_{11})| = 0\}, \tag{6}$$

$$\Sigma s \equiv \{(v_3, v_6, v_{10}, v_{11}) \in \Sigma;\ \ \text{real eigenvalues of } \partial G/\partial(v_{10}, v_{11}) \text{ are negative}\}, \tag{7}$$

$$\Sigma h \equiv \Sigma r \setminus \Sigma s, \tag{8}$$

$$\Sigma f \equiv \mathrm{Cl}\,\Sigma s \cap \mathrm{Cl}\,\Sigma h. \tag{9}$$

The eigenvalues of $\partial G/\partial(v_{10}, v_{11})$ at $p \in \Sigma$ are

$$\lambda = -(1/R_a + 1/R_c) \pm (f'(-v_3 - v_{10}) f'(-v_6 - v_{11}))^{1/2}. \tag{10}$$

Therfore, $p \in \Sigma s$ if and only if

$$f'(-v_3 - v_{10}) f'(-v_6 - v_{11}) \geqq 0 \text{ and}$$

$$f'(-v_3 - v_{10}) f'(-v_6 - v_{11}) < (1/R_a + 1/R_c)^2, \tag{11}$$

or

$$f'(-v_3 - v_{10}) f'(-v_6 - v_{11}) < 0 \quad (f' \text{ is the derivative of } f). \tag{12}$$

As a result,

$$\Sigma s = \{p \in \Sigma;\ |v_3 + v_{10}| > \alpha \text{ or } |v_6 + v_{11}| > \alpha\}, \tag{13}$$

$$\Sigma f = \{p \in \Sigma;\ |v_3 + v_{10}| = \alpha \text{ and } |v_6 + v_{11}| < \alpha, \text{ or } |v_3 + v_{10}| < \alpha \text{ and } |v_6 + v_{11}| = \alpha\}. \tag{14}$$

Let $\Pi_2 : R^4 \to R^2$,

$$(v_3, v_6, v_{10}, v_{11}) \to (v_3 + v_{10}, v_6 + v_{11}) \equiv (x, y), \tag{15}$$

then the following two notations are defined:

$$\Pi_2\Sigma f' \equiv \{(x, y) \in \Pi_2\Sigma f;\ x = \alpha\}, \tag{16}$$

$$\Pi_2\Sigma f'' \equiv \{(x, y) \in \Pi_2\Sigma f;\ y = \alpha\}. \tag{17}$$

R^2		$x=-\alpha$	$x=\alpha$	
	$\Pi_2\Sigma s$	$\Pi_2\Sigma s$	$\Pi_2\Sigma s$	
		$\Pi_2\Sigma f''$		$y=\alpha$
	$\Pi_2\Sigma s$	$\Pi_2\Sigma h$	$\Pi_2\Sigma f'$ $\Pi_2\Sigma s$	$y=-\alpha$
		$\Pi_2\Sigma s$	$\Pi_2\Sigma s$	

Moreover, we define D^{ij} $(i, j = \pm 0)$, $D^{00} = \Pi_2\Sigma h$, as the figure,

R^2		$x=-\alpha$	$x=\alpha$	
	D^{-+}	D^{0+}	D^{++}	$y=\alpha$
	D^{-0}	D^{00}	D^{+0}	
	D^{--}	D^{0-}	D^{+-}	$y=-\alpha$

and

$$\Sigma s^{ij} \equiv (\Pi_2^{-1} D^{ij}) \cap \Sigma s. \tag{18}$$

Next, we show analysis of $\Sigma f'$.
Let $(a_3, a_6, a_{10}, a_{11}) \in \Sigma f'$, then

$$
\begin{aligned}
&a_3+a_{10}=\alpha, \quad |a_6+a_{11}| < \alpha, \\
&-\alpha/R_c-\beta(a_6+a_{11})/\alpha+\beta-a_{10}/R_a=0, \qquad (19)\\
&-(a_6+a_{10})/R_c \qquad\qquad -a_{11}/R_a=0.
\end{aligned}
$$

Thus, $(a_3, a_6, a_{10}, a_{11}) \in \Sigma f'$ is parameterized by a_3 as follows:

$$
\begin{aligned}
&a_6 =(1/R_a+1/R_c)(a_3-\alpha)\alpha/\beta-\alpha(\alpha/\beta R_c-1)(R_a/R_c+1), \\
&a_{10}=\alpha-a_3, \qquad (20)\\
&a_{11}=(\alpha-a_3)\alpha/\beta R_c+\alpha^2 R_a/\beta R_c^2-\alpha R_a/R_c,
\end{aligned}
$$

with

$$\alpha(1+R_a/R_c)-2\beta R_a < a_3 < \alpha(1+R_a/R_c), \qquad (21)$$

(when $p=(\alpha, -\alpha)$, $a_3=\alpha(1+R_a/R_c)-2\beta R_a$ and when $p=(\alpha, \alpha)$, $a_3=\alpha(1+R_a/R_c)$, $q \in \Sigma f'$, $p=\Pi_2(q) \in \Pi_2 \Sigma f' \subset R^2$). For the above point in $\Sigma f'$, let

$$\Pi_2(a_3, a_6, a_{10}, a_{11})=(a_3+a_{10}, a_6+a_{11})=(x, y) \in \Pi_2 \Sigma f', \qquad (22)$$

then

$$x=\alpha, \quad y=\alpha a_3/\beta R_a-(1/R_a+1/R_c)\alpha^2/\beta+\alpha, \qquad (23)$$

with a_3 satisfying (21). Therefore, by (20), (23), $\Pi_2 | \Sigma f'$ is an embedding.

3. <u>The jumping orbits starting from $\Sigma f'$.</u>

For $q=(a_3, a_6, a_{10}, a_{11}) \in \Sigma f'$, we denote

$$Rq^2 \equiv \{(a_3, a_6, v_{10}, v_{11});\ v_{10}, v_{11} \in R^1\}, \qquad (24)$$

then

$$Cl\,\Sigma h \cap Rq^2=\{q\}. \qquad (25)$$

In fact, $(a_3, a_6, v_{10}, v_{11}) \in \Sigma h \cap Rq^2$, then

$$
\begin{aligned}
&(a_3+v_{10})/R_c+(a_6+v_{11})\beta/\alpha+v_{10}/R_a-\beta=0, \qquad (26)\\
&(a_6+a_{11})/R_c+(a_3+v_{11})\beta/\alpha+v_{11}/R_a-\beta=0.
\end{aligned}
$$

This equation (26) has a unique solution (v_{10}, v_{11}) by the assumption (4). Since,

$$q \in Cl\,\Sigma h \cap Rq^2, \qquad (27)$$

we have (25).

On the set $\Sigma s^{0+} \cap Rq^2$ $(q=(a_3, a_6, a_{10}, a_{11}) \in \Sigma f')$,
let $(a_3, a_6, v_{10}, v_{11}) \in \Sigma s^{0+} \cap Rq^2$, then

$$
\begin{aligned}
&(a_3+v_{10})/R_c+v_{10}/R_a=0, \qquad (28)\\
&(a_6+v_{11})/R_c+(a_3+v_{10})\beta/\alpha+v_{11}/R_a-\beta=0,
\end{aligned}
$$

therefore,

$$
\begin{aligned}
&v_{10}=-a_3 R_a/(R_a+R_c), \\
&v_{11}=(-R_a a_6+\beta R_a R_c)/(R_a+R_c)-R_a R_c^2 a_3 \beta/\alpha(R_a+R_c)^2. \qquad (29)
\end{aligned}
$$

Let $\Pi_2(\Sigma s^{0+} \cap Rq^2) \equiv (x, y) \in B \subset R^2$ $(x=a_3+v_{10}, y=a_6+v_{11})$, then

$$
\begin{aligned}
B:\ &x=R_c a_3/(R_a+R_c), \qquad (30)\\
&y=-R_a R_c^2 \beta a_3/\alpha(R_a+R_c)^2+R_c a_6/(R_a+R_c)+R_a R_c \beta/(R_a+R_c),
\end{aligned}
$$

and then $(\alpha, \alpha) \in \mathrm{Cl}B$.
On the set $\Sigma s^{-+} \cap Rq^2$, $(a_3, a_6, v_{10}, v_{11}) \in \Sigma s^{-+} \cap Rq^2$,

$$(a_3+v_{10})/R_c+v_{10}/R_a \quad =0,$$
$$(a_6+v_{11})/R_c+v_{11}/R_a-2\beta=0, \tag{31}$$

therefore,

$$v_{10}=-R_a a_3/(R_a+R_c), \quad v_{11}=(2\beta R_a R_c-R_a a_6)/(R_a+R_c). \tag{32}$$

Let $\Pi_2(\Sigma s^{-+} \cap Rq^2) \equiv (x, y) \in A \subset R^2$ $(x=a_3+v_{10}, y=a_6+v_{11})$, then

$$A: \quad x=R_c a_3/(R_a+R_c), \tag{33}$$
$$y=(2\beta R_a R_c+R_c a_6)/(R_a+R_c).$$

Since, a_6 is a linear function of a_3 by (20), the set A and set B are linear segments. Furthermore, for $x= R_c a_3/(R_a+R_c)=-\alpha$, the y's in (30) and (33) are equal, thus $\mathrm{Cl}A \cup \mathrm{Cl}B$ is cconnected. By the assumption (4) and (21), we have

$$R_a(\alpha/R-2\beta)<a_3<0<\alpha(1+R_a/R_c). \tag{34}$$

Let Γ_1 be the set of the points in the traces (fast orbits) starting from $\Sigma f'$. Then, Γ_1 is a 2-dimensional manifold with boundary (like a belt). We will show that $A \cup B \subset \partial\Gamma_1$ and that a trace in Γ_1 is a curve with end points in $\Sigma f'$ and $A \cup B$.

Put $D^{ij}{}_* \equiv \Pi_2{}^{-1} D^{ij} \subset R^4$ and $\Gamma_1{}^{ij} \equiv \Gamma_1 \cap D^{ij}{}_*$, then <u>the fast vector field Y^{oo}</u> on $\Gamma_1{}^{oo}$ is defined by

$$Y^{oo}: \quad \dot{v}_3=0, \quad \dot{v}_6=0,$$
$$\dot{v}_{10}=-(a_3+v_{10})/R_c-(a_6+v_{11})\beta/\alpha-v_{10}/R_a+\beta, \tag{35}$$
$$\dot{v}_{11}=-(a_6+v_{11})/R_c-(a_3+v_{10})\beta/\alpha-v_{11}/R_a+\beta,$$

for any fixed $(a_3, a_6, a_{10}, a_{11}) \in \Sigma f'$. By the map $\Pi_2 | \Gamma_1{}^{oo}: \Gamma_1{}^{oo} \to R^2$, Y^{oo} is induced to <u>a vetor field $(\Pi_2)_* Y^{oo}$</u> on $D^{oo} \subset R^2$ as follows:

$$(\Pi_2)_* Y^{oo}: \quad \dot{x}=-x/R-\beta y/\alpha+(a_3/R_a+\beta), \tag{36}$$
$$\dot{y}=-\beta x/\alpha-y/R+(a_6/R_a+\beta).$$

The orbit of (36) with an initial point $p=\Pi_2(q)=(a_3+a_{10}, a_6+a_{11})$ is the curve $Cq(k)$,

$$Cq(k)=(a_3+a_{10}-k, a_6+a_{11}+k)=(\alpha-k, a_6+a_{11}+k). \tag{37}$$

Because, by (35), (36),

$$\dot{x}+\dot{y}=-(\beta/\alpha+1/R)(x+y)+(a_3+a_6)/R_a+2\beta$$
$$=-(\beta/\alpha+1/R)(a_3+a_{10}+a_6+a_{11})+2\beta=\dot{v}_{10}+\dot{v}_{11}=0. \tag{38}$$

As an unstable direction at $q \in D^{oo} \cap \Sigma$ is $(v_{10}, v_{11})=(k, -k)$, it is a jumping direction at $p \in \Sigma f'$.

The above map $\Pi_2 | \Gamma_1{}^{oo}: \Gamma_1{}^{oo} \to D^{oo}$ is a diffeomorphism. Because, $q=(a_3, a_6, a_{10}, a_{11}) \in \Sigma f'$ is coordinated by a_3((20)), and locally, $\Gamma_1{}^{oo}$ is coordinated by (a_3, a_6) and any orbit of Y^{oo} is mapped onto an orbit of (36). This map of the orbit is regular at each point, by (35), (36). Moreover, it is diffeomorphic, since the curve $Cq(k)$ has no self-intersection and the vector

$(\dot{x}, \dot{y})$ is nonsingular at each point on $Cq(k)$. As Π_2: $\Sigma f' \rightarrow \Pi_2(\Sigma f')$, $q \in \Sigma f' \rightarrow (a_3+a_{10}, a_6+a_{11})$ is a diffeomorphism, if $q \neq q'$, then $Cq(k) \neq Cq'(k)$ and $Cq(k) \cap Cq'(k) = \phi$.

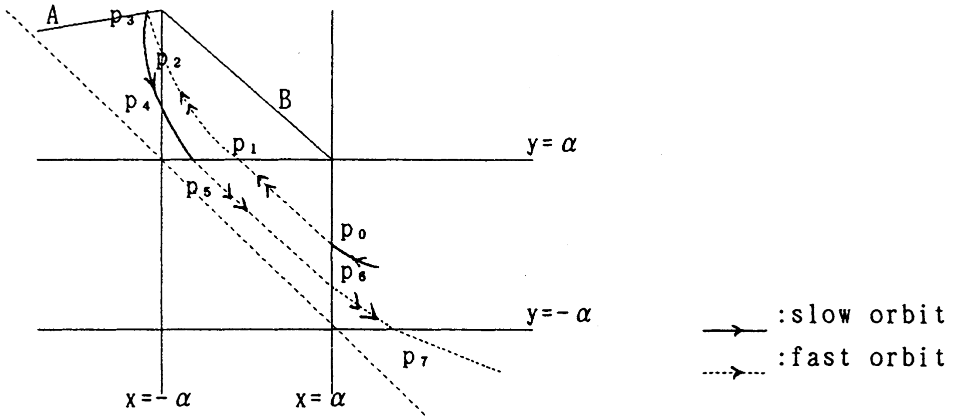
The fast vector field Y^{0+} on $\Gamma_1{}^{0+}$ and $(\Pi_2)_* Y^{0+}$ on $\Delta_B \subset D^{0+}$ are as follows:

$$Y^{0+}: \quad \dot{v}_3 = 0, \ \dot{v}_6 = 0, \quad \dot{v}_{10} = -(a_3+v_{10})/R_c - v_{10}/R_*, \quad \dot{v}_{11} = -(a_6+v_{11}) - (a_3+v_{10})\beta/\alpha - v_{11}/R_* + \beta, \tag{39}$$

$$(\Pi_2)_* Y^{0+}: \quad \dot{x} = -x/R + a_3/R_*, \quad \dot{y} = -\beta x/\alpha - y/R + a_6/R_* + \beta, \tag{40}$$

on Δ_B, where

$$\Delta_B = \{(x, y);\ \alpha < y < (\alpha/\beta R - R\beta/\alpha)x + (R\beta - \alpha^2/\beta R + \alpha) \text{ and } x+y > 0 \text{ and } -\alpha < x < \alpha\}. \tag{41}$$



The solution of (40) with the initial condition

$$(x(0), y(0)) = p_1 = (x_1, y_1), \quad x_1 = a_6 + a_{11}, \quad y_1 = \alpha, \tag{42}$$

is given as follows:

$$x(t) = (x_1 - Ra_3/R_*)e^{-t/R} + Ra_3/R_*,$$
$$y(t) = ((\beta^2R^2/\alpha^2 - R/R_*)a_6 + \alpha - \beta^2R^2/\alpha)e^{-t/R} + (\beta/\alpha)(-x_1 + Ra_3/R_*)te^{-t/R} - \beta R^2 a_3/\alpha R_* + R(a_6/R_* + \beta). \tag{43}$$

By (40), (43) and (21), on Δ_B,

$$\dot{x} = (a_3/R_* - x_1/R)e^{-t/R} < (\alpha - x_1)e^{-t/R}/R < 0, \quad (-\alpha < x_1 < \alpha). \tag{44}$$

By (43), (30),

$$\lim_{t \to \infty}(x(t), y(t)) = (Ra_3/R_*, -\beta R^2 a_3/\alpha R_* + R(a_6/R_* + \beta)) \in B. \tag{45}$$

By the property of Π_2 and (39), (40), any orbit of Y^{0+} starting from $p_1 = (x_1, \alpha)$ arrives at a point p_2:

$$\text{(i) } p_2 = (-\alpha, y_2) \in \Delta_B \quad \text{if } R_*(\alpha/R - 2\beta) < a_3 < -\alpha R_*/R, \tag{46}$$

$$\text{or (ii) } p_2 \in B \quad \text{if } -\alpha R_*/R < a_3 < \alpha R_*/R.$$

Similarly as $\Pi_2 \mid \Gamma_1$, $\Pi_2 \mid \Gamma_1{}^{0+}: \Gamma_1{}^{0+} \rightarrow \Delta_B$ is an orbit preserving diffeomorphism. The fast vector field Y^{-+} on $\Gamma_1{}^{-+}$ and $(\Pi_2)_* Y^{-+}$ on $\Delta_A \subset D^{-+}$

are as follows:

$$Y^{-+}: \quad \dot{v}_3=0,\ \dot{v}_6=0,$$

$$\dot{v}_{10}=-(a_3+v_{10})/R_c-v_{10}/R_a, \qquad (47)$$

$$\dot{v}_{11}=-(a_6+v_{11})/R_c-v_{11}/R_a+2\beta,$$

$$(\Pi_2)_*Y^{+-}: \quad \dot{x}=-x/R+a_3/R_a, \qquad (48)$$

$$\dot{y}=-y/R+a_6/R_a+2\beta,$$

on Δ_A, where

$$\Delta_A=\{(x,y);\ y<\alpha x/\beta R+(2R\beta-\alpha^2/\beta R+\alpha) \text{ and } x+y>0 \text{ and } x<-\alpha\}. \qquad (49)$$

The solution of (48) with the initial condition

$$(x(0),y(0))=p_2=(x_2,y_2),\quad x_2=-\alpha, \qquad (50)$$

is given as follows:

$$x(t)=-(\alpha+Ra_3/R_a)e^{-t/R}+Ra_3/R_a, \qquad (51)$$

$$y(t)=(y_2-Ra_6/R_a-2R\beta)e^{-t/R}+R(a_6/R_a+2\beta).$$

By (51) and (33), in the case of (i) $p_2=(-\alpha,y_2)\in\Delta_B$,

$$\lim_{t\to\infty}(x(t),y(t))=(Ra_3/R_a,R(a_6/R_a+2\beta))\in A. \qquad (52)$$

Similarly as $\Pi_2|\Gamma_1^{00}$ and $\Pi_2|\Gamma_1^{0+}$, $\Pi_2|\Gamma_1^{-+}:\Gamma_1^{-+}\to\Delta_A$ is an orbit preserving diffeomorphism. From the above discussions, we have proved the following Lemma1..

Lemma1.

Let Y be the fast vector field defined by

$$Y: \quad \dot{v}_3=0,\ \dot{v}_6=0, \qquad (53)$$

$$(\dot{v}_{10},\dot{v}_{11})=G(v_3,v_6,v_{10},v_{11}),$$

then the orbit of Y with the initial point $q_0=(a_3,a_6,a_{10},a_{11})\in\Sigma f'$ arrives at an equilibrium point (sink) of Y contained an arc $\underline{A}\cup\underline{B}$ in Σs, where $\underline{A}\equiv\Pi_2^{-1}(A)\cap\Sigma s^{-+}$, $\underline{B}\equiv\Pi_2^{-1}(B)\cap\Sigma^{0+}$.

4. The slow orbits in Σs.

The slow vector field X at $(v_3,v_6,v_{10},v_{11})\in\Sigma s$ is as follows:

$$X: \quad \dot{v}_3=-(v_3+v_{10})/CR_c,$$

$$\dot{v}_6=-(v_6+v_{11})/CR_c, \qquad (54)$$

$$\dot{v}_{10}=0,\ \dot{v}_{11}=0,$$

$$\Sigma s^{-+}: \quad \dot{v}_{10}=(v_3+v_{10})/R_c+v_{10}/R_a=0, \qquad (55)$$

$$\dot{v}_{11}=(v_6+v_{11})/R_c+v_{11}/R_a-2\beta=0$$

The tangent space $T\Sigma s^{-+}$ at any point in Σs^{-+} is as follows:

$$T\Sigma s^{-+}: \quad (dv_3+dv_{10})/R_c+(dv_{10})/R_a=0, \qquad (56)$$

$$(dv_6+dv_{11})/R_c+(dv_{11})/R_a=0.$$

For each $q=(b_3,b_6,b_{10},b_{11})\in\Sigma s$, let $\Pi_\Sigma:TqR^4\to Tq\Sigma s$ be the natural projection $R^4=Tq\Sigma s+TqRq^2\to Tq\Sigma s$, where $Rq^2\equiv\{(b_3,b_6,v_{10},v_{11});\ v_{10},v_{11}\in R^1\}$.

For $q=(v_3, v_6, v_{10}, v_{11}) \in \Sigma s^{-+}$,

$$Xq=(-1/CR_c)((v_3+v_{10}), (v_6+v_{11}), 0, 0), \tag{57}$$

we have the followings: put $\Pi_\Sigma Xq^{-+}=(x_3, x_6, x_{10}, x_{11})$, then there is a point $(0, 0, y_{10}, y_{11}) \in TqRq^2$ satisfying

$$\begin{pmatrix} x_3 \\ x_6 \\ x_{10} \\ x_{11} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ y_{10} \\ y_{11} \end{pmatrix} + \begin{pmatrix} -(v_3+v_{10})/CR_c \\ -(v_6+v_{11})/CR_c \\ 0 \\ 0 \end{pmatrix}, \tag{58}$$

hence, by (56),

$$\Pi_\Sigma Xq^{-+} = (1/CR_c) \begin{pmatrix} -(v_3+v_{10}) \\ -(v_6+v_{11}) \\ (v_3+v_{10})R_a/(R_a+R_c) \\ (v_6+v_{11})R_a/(R_a+R_c) \end{pmatrix}. \tag{59}$$

On the other hand,

$$\Pi_2 \begin{pmatrix} v_3 \\ v_6 \\ v_{10} \\ v_{11} \end{pmatrix} = \begin{pmatrix} v_3+v_{10} \\ v_6+v_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_3 \\ v_6 \\ v_{10} \\ v_{11} \end{pmatrix}, \tag{60}$$

hence

$$(\Pi_2)_* \begin{pmatrix} dv_3 \\ dv_6 \end{pmatrix} = \begin{pmatrix} dv_3+dv_{10} \\ dv_6+dv_{11} \end{pmatrix}. \tag{61}$$

By the map $\Pi_2 \mid \Sigma s^{IJ}: \Sigma s^{IJ} \rightarrow D^{IJ}$ (diffeomorphism),

$$(\Pi_2)_* \Pi_\Sigma Xq^{-+} = (1/CR_c) \begin{pmatrix} -(v_3+v_{10})+(v_3+v_{10})R_a/(R_a+R_c) \\ -(v_6+v_{11})+(v_6+v_{11})R_a/(R_a+R_c) \end{pmatrix}$$

$$= (-1/C(R_a+R_c)) \begin{pmatrix} v_3+v_{10} \\ v_6+v_{11} \end{pmatrix}, \tag{62}$$

therefore,

$$(\Pi_2)_* \Pi_\Sigma Xq^{-+}: \dot{x} = -x/C(R_a+R_c), \quad \dot{y} = -y/C(R_a+R_c), \tag{63}$$

and then a time scaled equation of (63) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = - \begin{pmatrix} x \\ y \end{pmatrix} \tag{64}$$

The solution of (64) with the initial condition $(x(0), y(0))=p_3=(x_3, y_3) \in A$ is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_3 e^{-t} \\ y_3 e^{-t} \end{pmatrix} \tag{65}$$

From (20) and (33),

$$(x_3, y_3)=(a_3R_c/(R_a+R_c), \alpha a^3/\beta Ra-\alpha^2/\beta R+2\beta R+\alpha). \tag{66}$$

There is a $t>0$ such that $(x(t), y(t))=p_4=(-\alpha, y_4) \in (\partial\Delta_A) \cap (\partial\Delta_B)$. In fact,

$$x_4=x(t)=a_3(R_c/(R_a+R_c))e^{-t}=-\alpha, \quad (e^{-t}=-\alpha R_a/Ra_3), \tag{67}$$

$$y_4=y(t)=-\alpha^2/\beta R+(\alpha^3/\beta R2-2\alpha\beta-\alpha^2/R)R_a/a_3.$$

In Σs^{0+},

$$\Sigma s^{0+}: \quad \dot v_{10} = (v_3+v_{10})/R_c+v_{10}/R_a=0, \tag{68}$$
$$\dot v_{11} = (v_6+v_{11})/R_c + \beta(v_3+v_{10})/\alpha + v_{11}/R_a - \beta = 0.$$

Similarly as in Σs^{-+}, we have, for $q=(v_3, v_6, v_{10}, v_{11}) \in \Sigma s^{0+}$,

$$\Pi_\Sigma X q^{0+} = (1/CR_c) \begin{pmatrix} -(v_3+v_{10}) \\ -(v_6+v_{11}) \\ (v_3+v_{10})R_a/(R_a+R_c) \\ (v_3+v_{10})\beta R^2/\alpha R_a+(v_6+v_{11})R_a/(R_a+R_c) \end{pmatrix}. \tag{69}$$

$$(\Pi_2)_* \Pi_\Sigma X q^{0+}: \quad \dot x = -x/C(R_a+R_c) \tag{70}$$
$$\dot y = (\beta Rx/\alpha - y)/C(R_a+R_c).$$

The solution of (70) with the initial condition $(x(0), y(0))=p_4=(x_4, y_4)=(-\alpha, y_4)$ is the followings:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -\alpha e^{-t} \\ -\beta Rte^{-t}+y_4e^{-t} \end{pmatrix}. \tag{71}$$

There is a $t>0$ such that $(x(t), y(t))=p_5=(x_5, y_5) \in \partial D^{00} \cap \partial D^{0+}$, since $y_4 > \alpha$ and $y_5=\alpha$, by (71) $y(t)\to 0$ $(t\to\infty)$.

<u>Lemma2.</u>

For the point $p_1=(x_1, \alpha)$, $p_5=(x_5, \alpha) \in \partial D^{00} \cap \partial D^{0+}$ defined above, we have $-\alpha < x_5 < x_1 < \alpha$.

(proof)

$(x_0, y_0)=p_0=\Pi_2(q_0)=\Pi_2(a_3, a_6, a_{10}, a_{11})=(a_3+a_{10}, a_6+a_{11})$, since $q_0 \in \Sigma f'$, $x_0=a_3+a_{10}=\alpha$. By applying the case of Σs^{-+}, starting from Σs^{+0},

$$y_0=a_6+a_{11}=\alpha a_3/\beta R_a - \alpha^2/\beta R+\alpha, \tag{72}$$

and by using $p_1=(x_1, y_1)=(y_0, x_0)$,

$$x_1=\alpha a_3/\beta R_a - \alpha^2/\beta R+\alpha. \tag{73}$$

Since $y_5=\alpha$, it follows that (71) implies

$$-\beta Rt+y_4=\alpha e^t. \tag{74}$$

By using $e^t=1+\theta t$ $(\theta>0)$, the solution of (74) is

$$t=(y_4-\alpha)/(\alpha\theta+\beta R). \tag{75}$$

From the assumption (4) and (21),

$$R_a(\alpha/R-2\beta)<a_3, \quad (\alpha/\beta(\alpha/R-2\beta)<\alpha a_3/\beta R_a, \quad R/(\alpha-2\beta R)/>R_a/a_3). \tag{76}$$

From (75), (76) and (73),

$$x_5 = -\alpha/e^{-t} = -\alpha/(1+\theta(y_4-\alpha)/(\alpha\theta+\beta R)), \tag{77}$$

$$\begin{aligned} x_1-x_5 &= \alpha a_3/\beta R_a - \alpha^2/\beta R+\alpha+\alpha/(1+\theta(y_4-\alpha)/(\alpha\theta+\beta R)) \\ &> \alpha(\alpha/R-2\beta)/\beta - \alpha^2/\beta R+\alpha+ \\ &\alpha(\alpha\theta+\beta R)/(\alpha\theta+\beta R+\theta(-\alpha^2/\beta R+(\alpha^3/\beta R^2-2\alpha\beta-\alpha^2/R)R/(\alpha-2\beta R)-\alpha) \\ &= -\alpha+\alpha = 0. \end{aligned} \tag{78}$$

□

5. A quasi-potential on the slow orbits.

According to the translation of the variables:

$$x \equiv (v_3+v_{10})/\alpha,\quad y \equiv (v_6+v_{11})/\alpha, \tag{79}$$

v_3, v_6, v_{10}, v_{11} and $\partial\Sigma$ are represented by the new coordinate as follows:

$$v_3=\alpha R_a x/R-R_a f(-\alpha y),\quad v_6=\alpha R_a y/R-R_a f(-\alpha x), \tag{80}$$

$$v_{10}=-\alpha R_a x/R_c+R_a f(-\alpha y),\quad v_{11}=-\alpha R_a y/R+R_a f(-\alpha x) \text{ and}$$

$$\partial\Sigma=\{(x,y)\in R^2;\ \begin{vmatrix} 1/R & -f'(-\alpha y) \\ -f'(-\alpha x) & 1/R \end{vmatrix}=0\}. \tag{81}$$

It follows that on the slow manifold ([1]):

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = -R/CR_cR_a(1-K^2h'(-x)h'(-y)) \begin{pmatrix} 1 & Kh'(-y) \\ Kh'(-x) & 1 \end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}, \tag{82}$$

$$f(v)=\beta(1+h(v/\alpha)), \tag{83}$$

and h is an odd function satisfying $\lim_{x\to\infty} h(x)=1$ and $h'(0)=1$. Then,

$$\partial\Sigma=\{(x,y)\in R^2;\ K^2h'(-x)h'(-y)=1\}. \tag{84}$$

The stable manifold S and the unstable manifold T at the equilibrium point $(x,y)=(0,0)$ in (82) are given by the diagonal subspaces

$$S=\{(x,y)\in R^2;\ x-y=0\} \text{ and } T=\{(x,y)\in R^2;\ x+y=0\}. \tag{85}$$

Introducing a Nishino-Tchizawa quasi-potential F ([1]):

$$F(x,y,K)=R_a\beta x/K-R_a f(-\alpha y)-R_a\beta y/K+R_a f(-\alpha x), \tag{86}$$

where $K\equiv R\beta/\alpha$ $(K>1)$, we can describe the convergence of the trajectories in the system to an attractor-inf limit cycle.

Theorem.

The slow orbit of the dimensionless equations (82) has a limit cycle L such that $L\subset T$ as an attractor.

(proof)

As $x_1>x_5$ is shown by Lemmwa 2., from the form of the quasi-potential F, we can conclude the values of F on the cross section $\partial\Sigma$ are monotone. Therefore, the values are convergent to the minimum where $(x,y)=(-\alpha,\alpha)$, or the maximum where $(x,y)=(\alpha,\alpha)$ on the compact subset $\partial\Sigma(-\alpha\leqq x\leqq\alpha)$. Furthermore, as $F(p_0)=F(p_3)$, $F(p_5)=F(p_8)$, ..., we can similarly see the convergence to the accumulation point $p_*(p^*)\in T\cap\Gamma$ on the other cross section Γ, which is the set of the arrival points after jumping, where $p_*\equiv\lim_{n\to\infty} p_n$ $((F(p_n)>F(p_{n+1}))$, $p^*\equiv\lim_{m\to\infty} p_m$ $((Fp_m)<F(p_{m+1}))$.

Thus, there is a limit cycle $L: L=\{(x,y)\in T;\ F(p_*)\leqq F(x,y)\leqq F(p^*)\}$.

□

Reference.

[1]: "Simulation of an Electronic Multivibrator", P.Ashwin, G.P.King, J.Nijhof, G.Ikegami and K.Tchizawa (to appear) (1993)