数理解析研究所講究録
第 878 巻 1994 年 51-55

Moduli space and Complex analytic Gel'fand Fuks cohomology of Riemann surfaces.

NARIYA KAWAZUMI*

Abstract. Let $C^{\times}$ be a once punctured compact Riemann surface and $L(C^{\times})$ the Lie algebra consisting of all complex analytic vectors on $C^{\times}$. We determine the q-th cohomology group of $L(C^{\times})$ with values in the complex analytic quadratic differentials on the p-fold product space $(C^{\times})^p$ for the case $p \geq q$. The cohomology group vanishes for $p > q$, and, for $p = q$, it forms a trivial constant sheaf on the dressed moduli $M_{g,\rho}$ of compact Riemann surfaces of genus g. (Furthermore the stalk does not depend on the genus g.) This induces a natural map of the cohomology group for $p = q$ into the (p,p) cohomology of the moduli $M_{g,\rho}$. We prove the map is a stable isomorphism onto the subalgebra generated by the Morita Mumford classes $e_n = \kappa_n$'s, which gives an affirmative evidence for the conjecture: the stable cohomology algebra of the moduli of compact Riemann surfaces would be generated by the Morita Mumford classes.

Let M_g denote the moduli space of compact Riemann surfaces of genus $g \geq 2$, i.e., the space consisting of all isomorphism classes of complex structures defined on the closed orientable C^∞ surface of genus g, Σ_g. It is a $3g-3$-dimensional complex analytic orbifold. It is valuable for topologists to investigate the space M_g because of the isomorphisms

$$\begin{aligned} H^*(M_g;\mathbb{Q}) &= H^*(B\,\mathrm{Diff}^+\,\Sigma_g;\mathbb{Q}) \\ &= \{\text{the rational characteristic classes of oriented } \Sigma_g \text{ bundles.}\} \end{aligned}$$

The stability theorem of Harer [H] asserts that the q-th rational cohomology group $H^q(M_g;\mathbb{Q})$ does not depend on the genus g, provided that $q < g/3$. It enables us to consider the stable cohomology algebra of the moduli spaces of compact Riemann surfaces $\lim_{g\to\infty} H^*(M_g;\mathbb{Q})$. In view of a theorem established by Morita [Mo] and Miller [Mi] independently, a polynomial algebra in countably many generators denoted by e_n, $n \in \mathbb{N}_{\geq 1}$, is embedded into this stable cohomology algebra:

$$\mathbb{Q}[e_n;\, n \geq 1] \hookrightarrow \lim_{g\to\infty} H^*(M_g;\mathbb{Q}). \tag{1}$$

*Department of Mathematics, Faculty of Science, University of Tokyo, Hongo, Tokyo, 113 Japan

1991 Mathematical Subject Classification. Primary 14H15. Secondary 57R32.

Here the class $e_n \in H^{2n}(M_g;\mathbb{Q})$ (or $H^{n,n}(M_g)$) is the n-th Morita Mumford class (the n-th tautological class) defined as follows. Let $C_g \to M_g$ be the universal family of compact Riemann surface of genus g and denote $e := c_1(T_{C_g/M_g}) \in H^2(C_g;\mathbb{Q})$ (or $H^{1,1}(C_g)$). Applying the fiber integral to the power of the class e, one defines

$$(2) \qquad e_n := \int_{\text{fiber}} e^{n+1} \in H^{2n}(M_g;\mathbb{Q}) \quad \text{or} \quad H^{n,n}(M_g)$$

[Mo][Mu]. Our purpose is to give a result which suggests us that the map (1) should be an isomorphism.

The theory of complex vector bundles is an ideal type for theories of other fiber bundles. The Grassmannian manifold, which classifies complex vector bundles, is a homogeneous space and its cohomology is described by cohomologies of Lie algebras. It suggests us that our moduli space M_g would be endowed with some homogeneous structure under suitable modifications, and that its cohomology would be described through the homogeneous structure.

As was observed by Kontsevich [Ko] and Beilinson - Manin - Schechtman [BMS], the dressed moduli space $M_{g,\rho}$ has an infinitesimal homogeneous structure. More precisely, there exists a Lie algebra homomorphism of a certain Lie algebra $\mathfrak{d}_\rho$ to the Lie algebra of complex analytic vector fields on $M_{g,\rho}$, and the tangent space at each point of $M_{g,\rho}$ is spanned by vectors coming from the algebra $\mathfrak{d}_\rho$. From the Harer stability [H] the dressed moduli $M_{g,\rho}$ has the same stable cohomology as that of the moduli M_g.

Let $\rho > 0$ be a positive real number. Following Arbarello, DeConcini, Kac and Procesi [ADKP], we define the dressed moduli $M_{g,\rho}$ as the space consisting of all triples (C,p,z) , where C is a compact Riemann surface of genus g, p is a point of C, and z is a complex coordinate of a neighbourhood U of p satisfying the conditions

$$z(p) = 0 \quad \text{and} \quad z(U) \supset \{z \in \mathbb{C}; |z| \leq \rho\}.$$

The space $M_{g,\rho}$ is acted on by the Lie algebra $\mathfrak{d}_\rho$ consisting of all complex analytic vector fields on the punctured disk $\{z \in \mathbb{C}; 0 < |z| \leq \rho\}$ through moving the glueing map between $C - \{p\}$ and $\{|z| \leq \rho\}$. This action is transitive and the isotropy subalgebra $(\mathfrak{d}_\rho)_x$ at each point $x = (C,p,z) \in M_{g,\rho}$ is equal to the Lie algebra consisting of all complex analytic vector fields on the open Riemann surface $C - \{p\}$. Especially the cotangent space $T_x^*M_{g,\rho}$ is equal to the space consisting of all complex analytic quadratic differentials on the open Riemann surface $C - \{|z| \leq \rho\}$.

We need compute the cohomology group of the isotropy subalgebra $(\mathfrak{d}_\rho)_x$ with coefficients in the n-th cotangent space $\bigwedge^n T_x^* M_{g,\rho}$. It has a natural correspondence to the cohomology group of the dressed moduli $M_{g,\rho}$. Thus we need construct some general theories on cohomology groups of the Lie algebra of complex analytic vector fields on open Riemann surfaces U with coefficients in the tensors on the product space U^n, i.e., the tensor-valued complex analytic Gel'fand-Fuks cohomology theory for open Riemann surfaces.

The paper [Ka] is devoted to the basic part of this general theory. For an open Riemann surface U and a finite subset $S \subset U$ we denote by $L(U, S)$ the topological Lie algebra consisting of all complex analytic vector fields on U which have zeroes at all points in S. In a classical way the computation of the cohomology of $L(U, S)$ with coefficients in the *global* tensor fields is reduced to those of the cohomologies with coefficients in the *germs* of the tensors. The main result of [Ka] asserts that the cohomology of $L(U, S)$ with coefficients in the germs of tensors decomposes itself into the global part derived from the homology of U and the local part coming from the coefficients. Its proof is obtained by translating the Bott-Segal addition theorem of the C^∞ Gel'fand-Fuks cohomology [BS] into our complex analytic situation. Thus the computation is reduced to that in the case when $(U, S) = (\mathbb{C}, \{0\})$ and when the coefficients is the tensor fields on $\mathbb{C}^n$

It is this case that we investigate in the paper [Ka1]. Then the difficulties concentrate themselves into the origin $0 \in \mathbb{C}^n$. Now we shall eliminate the origin. The residual terms which arise from the elimination can be controlled through a generalization by Scheja of the second Riemann (Hartogs) continuation theorem [S]. In other words, the elimination induces a cohomology exact seqence of the Lie algebra $L(\mathbb{C}, \{0\})$ with coefficients in the tensors on $\mathbb{C}^n$. As an application we obtain an explicit description of the cohomology algebra of $L(\mathbb{C})$ with coefficients in the complex analytic functions on $\mathbb{C}^n$.

Now we return to the dressed moduli $M_{g,\rho}$. By the result in [Ka] the computation of the cohomology group $H^q((\mathfrak{d}_\rho)_x; \bigwedge^p T_x^* M_{g,\rho})$ for each point $x \in M_{g,\rho}$ is reduced to that of the group $H^q(W_1; \bigwedge^p Q)$, where $W_1 := L(\mathbb{C}, \emptyset)$ and Q is the W_1 module of complex analytic quadratic differentials on $\mathbb{C}$. From the cohomology exact sequence in [Ka1] follows $H^q(W_1; \bigwedge^p Q) = 0$ and so $H^q((\mathfrak{d}_\rho)_x; \bigwedge^p T_x^* M_{g,\rho}) = 0$ for $q < p$.

In [Ka2] we reconstruct the Chern class e of the relative tangent bundle T_{C_g/M_g} and the fiber integral of the power e^{n+1}, $\kappa_n \in H^n((\mathfrak{d}_\rho)_x; \bigwedge^n T_x^* M_{g,\rho})$, under our Lie algebro-cohomological framework. We prove that the algebra $\bigoplus_{p\geq 0} H^p((\mathfrak{d}_\rho)_x; \bigwedge^n T_x^* M_{g,\rho})$ is generated by these classes κ_n's, $n \geq 1$, and that the class κ_n has a natural correpondence to the

n-th Morita-Mumford class $e_n \in H^{n,n}(M_{g,\rho})$. It follows from the Miller-Morita theorem stated above that the classes κ_n's are algebraically independent. Consequently we obtain

THEOREM. *For each point $x \in M_{g,\rho}$ we have $H^q((\mathfrak{d}_\rho)_x; \bigwedge^p T_x^* M_{g,\rho}) = 0$, if $p > q$, and*

$$\bigoplus_{p\geq 0} H^p((\mathfrak{d}_\rho)_x; \bigwedge^p T_x^* M_{g,\rho}) = \mathbb{C}[\kappa_n; n \geq 1].$$

The sheaf of germs of n-forms on $M_{g,\rho}$, $\Omega^n_{M_{g,\rho}}$, is a sheaf of $\mathfrak{d}_\rho$ modules. Hence the standard complex $C^*(\mathfrak{d}_\rho; \Omega^n_{M_{g,\rho}})$ of the Lie algebra $\mathfrak{d}_\rho$ is a cochain complex of sheaves on $M_{g,\rho}$. We denote by $H^*(\mathfrak{d}_\rho : \Omega^n_{M_{g,\rho}})$ the cohomology sheaf on $M_{g,\rho}$ of this cochain complex. Taking into consideration a geometric interpretation of the Frobenius reciprocity law [B], we put a general hypothesis:

HYPOTHESIS.

$$H^*(\mathfrak{d}_\rho; \Omega^n_{M_{g,\rho}})_x = H^*((\mathfrak{d}_\rho)_x; \bigwedge^n T_x^* M_{g,\rho}) \quad (\forall x \in M_{g,\rho})$$

The hypothesis seems to be true. Although the author has no proof of this assertion at present, he believes that it shall be justified under some suitable modifications. So we denote by $H^{n,*}_{\mathfrak{d}_\rho}(M_{g,\rho})$ the hypercohomology of the cochain complex of sheaves $C^*(\mathfrak{d}_\rho; \Omega^n_{M_{g,\rho}})$ and call it *the $\mathfrak{d}_\rho$-equivariant $(n,*)$-cohomology of $M_{g,\rho}$*. Our results are reformulated as follows.

COROLLARY. *Under the above hypothesis we have*

$$\bigoplus_{p\geq q} H^{p,q}_{\mathfrak{d}_\rho}(M_{g,\rho}) = \mathbb{C}[e_n; n \geq 1].$$

This suggests that it is reasonable to conjecture that the stable cohomology algebra of the moduli of compact Riemann surfaces is generated by the Morita-Mumford classes e_n's.

REFERENCES

[ADKP] E. Arbarello, C. DeContini, V.G. Kac, and C. Procesi, *Moduli spaces of curves and representation theory*, Commun. Math. Phys. **117** (1988), 1–36.

[BMS] A. A. Beilinson, Yu. I. Manin and V. V. Schechtman, *Sheaves of Virasoro and Neveu-Schwartz algebras*, Lect. Note in Math. **1289** (1987), Springer, Berlin - Heidelberg - New York.

[B] R. Bott, *Homogeneous vector bundles*, Ann. Math. **66** (1957), 203–248.

[BS] R. Bott and G. Segal, *The cohomology of the vector fields on a manifold*, Topology **16** (1977), 285–298.

[H] J. Harer, *Stability of the homology of the mapping class group of orientable surfaces*, Ann. Math. **121** (1985), 215–249.

[Ka] N. Kawazumi, *On the complex analytic Gel'fand-Fuks cohomology of open Riemann surfaces*, Anal. Inst. Fourier (to appear).

[Ka1] ——————, *An application of the second Riemann continuation theorem to cohomology of the Lie algebra of vector fields on the complex line*, (preprint UTYO-MATH 93-18).

[Ka2] ——————, *Moduli space and complex analytic Gel'fand Fuks cohomology of Riemann surfaces*, (preprint UTYO-MATH 93-29).

[Ko] M.L. Kontsevich, *Virasoro algebra and Teichmüller spaces*, Functional Anal. Appl. **21** (1987), 156–157.

[Mi] E.Y. Miller, *The homology of the mapping class group*, J. Diff. Geom. **24** (1986), 1–14.

[Mo] S. Morita, *Characteristic classes of surface bundles*, Inventiones math. **90** (1987), 551–577.

[Mu] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and Geometry., Progr. Math. **36** (1983), 271–328.

[S] G. Scheja, *Riemannsche Hebbarkeitssätze für Cohomologieklassen*, Math. Ann. **144** (1961), 345–360.