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Extrapolation Methods for Large Systems of Ordinary Differential Equations

(常微分方程式の大きなシステムに対する補外法の有効性)

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Introduction

The finite element approximation of elastic initial-boundary value problems results in solving initial value problems for very large systems of ordinary differential equations. In the stepwise integration of these problems, the discretization of the time derivatives has to be carefully treated, since in the practical computation several hundreds or thousands steps are necessary to obtain useful information and in such process, the accumulation of truncation error may have serious influence on the results.

The purpose of the present paper is to propose the use of an extrapolation technique to improve the accuracy of the time-discretization without adding extra computation time so much. In particular, it is shown that an extrapolation applied to the one step *central difference* (CD) scheme (the so-called $\beta-$scheme [4], choosing $\beta = 0$), which is called the *extrapolated central difference* (ECD) scheme in this paper, is most practical both in saving the computing time and in improving the accuracy.

The application of the technique of extrapolation to ODE's has a long history and a lot of research papers which indicate its efficiency has been published. We refer, for example, to D.C.Joice survey paper [5] for details and its references. Nevertheless, as far as the authors know, it is not so widely utilized in the actual computation of very large systems in science and engineering. This is probably due to that these research do not necessarily take the size or special character of the systems into account. As a consequence, the interest is limited mainly to the accuracy and the discussion has been directed to the behavior of the extrapolation as the extrapolation step size tends to zero. However, for very large systems, what is important is the balance between the accuracy and the computing time.

The idea of the extrapolation is simple. Consider a one step method to solve an initial value problem. Let Δt be the basic step size and y_Δ be the approximate value obtained by using the step size $\Delta = \Delta t, \Delta t/2, \Delta t/4, \cdots$. Assume that the error permit an expansion of the form

$$y_\Delta = y_1 + a_1\Delta^2 + a_2\Delta^4 + \cdots.$$

We seek an approximate value $\bar{y}_1$ for y_1 from the equations

$$y_{\Delta t} = \bar{y}_1 + a_1\Delta t^2$$

$$y_{\Delta t/2} = \bar{y}_1 + a_1(\frac{1}{2}\Delta t)^2$$

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Then, since

$$\bar{y_1} = \frac{1}{3}(4y_{\Delta t/2} - y_{\Delta t}) = y_1 + O(\Delta t^4)$$

the error reduces to $O(\Delta t^4)$. The value $\bar{y_1}$ is the approximate value obtained in one extrapolation.

If we use the approximate value $y_{\Delta t/4}$ and add the $O(\Delta)$ term, then we can get a higher extrapolation with higher accuracy. As is seen in §5, the higher order extrapolation seems to have no practical use for very large systems due to too much computation time. However one extrapolation applied to the CD scheme has the same accuracy as the fourth order Runge-Kutta method and the computing time is about twice that of CD.

In Section 1 we introduce a one-step integration scheme equivalent to the standard CD scheme, which is known as the $\beta-$ scheme. In Section 2 the simplest extrapolation formula for the CD scheme is derived. Section 3 is devoted to give a difference expression for the solution of the ECD scheme. In Section 4 we give a sufficient condition to ensure the stability of the ECD scheme in the sense of energy. Finally in Section 5 we present some numerical results which show the practical efficiency of the ECD scheme.

In the actual computation the algorithm of ECD is slightly modified, so as to save the computing time. Thanks to this modification, the computing time of ECD scheme becomes $1.3 \sim 1.4$ times that of CD, since the restriction on the time mesh Δt is relaxed in ECD. The modified version of ECD is called MECD.

In this paper we treat only linear systems to examine the basic properties of the extrapolation in detail. However, this technique is applicable to non-linear problems too and our results will give useful suggestions for such cases.

§1. Derivation of one-step scheme equivalent to the central difference scheme

Consider an initial value problem for N ordinary differential equations

$$My'' + Ky = f(t) \tag{1.1}$$

derived from a finite element approximation of the equation of motion to describe a linear elastic vibration. Here M and K are the mass and stiffness matrices, respectively. We assume that M is diagonal and K is symmetric and positive definite. The right-hand side of (1.1) is assumed to be zero in this paper, since this term has no essential influence on the following discussion.

We also assume the inverse inequality [2]

$$\|y\|_K \le \frac{C_0}{h}\|y\|_M \qquad (\|y\|_K^2 = (Ky, y)) \tag{1.2}$$

where h is a parameter to denote the size of the finite element. In order to apply the extrapolation technique we rewrite the central difference scheme to a one step scheme known as "$\beta-$ scheme". Introduce $A = -M^{-1}K$ to write (1.1) as

$$y'' = Ay. \tag{1.3}$$

The central difference approximation of this system is

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta t^2} = Ay_i \qquad (i = 1, 2, \cdots). \tag{1.4}$$

Introduce $\{z_i\}$ $(i = 1, 2, \cdots)$ by $z_0 = y'(0)$

$$z_{i+1} = z_i + \frac{1}{2}\Delta t A(y_{i+1} + y_i) \qquad (i = 0, 1, 2, \cdots). \tag{1.5}$$

Theorem 1 *The sequence $\{y_i\}$ determined by (1.4) satisfies*

$$(1.6) \qquad y_{i+1} = y_i + \Delta t z_i + \frac{1}{2}\Delta t^2 A y_i \qquad (i = 1, 2, \cdots),$$

provided y_1 is given by

$$(1.7) \qquad y_1 = y_0 + \Delta t z_0 + \frac{1}{2}\Delta t^2 A y_0. \qquad \square$$

Substitution of (1.6) into (1.5) leads to the equation

$$z_{i+1} = \frac{1}{2}\Delta t A(2I + \frac{1}{2}\Delta t^2 A)y_i + (I + \frac{1}{2}\Delta t^2 A)z_i.$$

Therefore the central difference equation (1.4) is written in matrix form as

$$(1.8) \qquad \begin{pmatrix} y_{i+1} \\ z_{i+1} \end{pmatrix} = Q_{\Delta t}\begin{pmatrix} y_i \\ z_i \end{pmatrix}, \qquad Q_{\Delta t} = \begin{pmatrix} I + \frac{1}{2}\Delta t^2 A & \Delta t I \\ \frac{1}{2}\Delta t A(2I + \frac{1}{2}\Delta t^2 A) & I + \frac{1}{2}\Delta t^2 A \end{pmatrix},$$

provided y_1 is determined by (1.7). The computation proceeds as follows.

$$\begin{aligned} z_0 &= y'(0) \\ y_{i+1} &= (I + \frac{1}{2}\Delta t^2 A)y_i + \Delta t z_i \qquad i = 0, 1, 2, \cdots \\ z_{i+1} &= z_i + \frac{1}{2}\Delta t A(y_{i+1} + y_i). \end{aligned}$$

Note that the computation of the type Ay_i, which includes the computation of the stiffness matrix K, appears only one time at each step if the vector Ay_i is stored.

§2. Extrapolation of the central difference scheme.

In order to derive an extrapolation formula we have to know the asymptotic expansion of the error. In our case, however, the extrapolation formula is easily determined.

Assume that $U_0 = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$ is given as an initial value of the CD scheme and let $\bar{U}_1 = \begin{pmatrix} \bar{y}_1 \\ \bar{z}_1 \end{pmatrix}$ and $\bar{\bar{U}}_1 = \begin{pmatrix} \bar{\bar{y}}_1 \\ \bar{\bar{z}}_1 \end{pmatrix}$ be the values after Δt with step-size Δt and $\Delta t/2$, respectively, that is,

$$(2.1) \qquad \bar{U}_1 = Q_{\Delta t} V_0, \qquad \bar{\bar{U}}_1 = Q^2_{\Delta t/2} V_0,$$

$$Q^2_{\Delta t/2} = \begin{pmatrix} I + \frac{1}{2}\Delta t^2 A + \frac{1}{32}\Delta t^4 A^2 & \Delta t I + \frac{1}{8}\Delta t^3 A \\ \Delta t A + \frac{3}{16}\Delta t^3 A^2 + \frac{1}{128}\Delta t^5 A^3 & I + \frac{1}{2}\Delta t^2 A + \frac{1}{32}\Delta t^4 A^2 \end{pmatrix}.$$

Taking the relations $z_0 = y'(0)$ and $y'' = Ay$ into account we have, from equation (2.1),

$$\bar{y}_1 = y_0 + \frac{1}{2}\Delta t^2 A y_0 + \Delta t z_0 = y_0 + \Delta t y'(0) + \frac{1}{2}\Delta t^2 y''(0)$$

$$\begin{aligned} \bar{\bar{y}}_1 &= y_0 + \frac{1}{2}\Delta t^2 A y_0 + \frac{1}{32}\Delta t^4 A^2 y_0 + \Delta t z_0 + \frac{1}{8}\Delta t^3 A z_0 \\ &= y_0 + \Delta t y_0' + \frac{1}{2}\Delta t^2 y_0'' + \frac{1}{8}\Delta t^3 y_0^{(3)} + \frac{1}{32}\Delta t^4 y_0^{(4)}, \end{aligned}$$

where $y_0^{(k)} = \dfrac{d^k y(0)}{dt^k}$.
Therefore hold

$$\text{(2.2)} \qquad \begin{cases} \bar{y}_1 - y(\Delta t) = -\frac{1}{6}\Delta t^3 y_0^{(3)} - \frac{1}{24}\Delta t^4 y_0^{(4)} + O(\Delta t^5) \\ \bar{\bar{y}}_1 - y(\Delta t) = -\frac{1}{24}\Delta t^3 y_0^{(3)} - \frac{1}{96}\Delta t^4 y_0^{(4)} + O(\Delta t^5). \end{cases}$$

Similarly we have

$$\begin{cases} \bar{z}_1 = z_0 + \Delta t y_0'' + \frac{1}{2}\Delta t^2 y_0^{(3)} + \frac{1}{4}\Delta t^3 y_0^{(4)} \\ \bar{\bar{z}}_1 = z_0 + \Delta t y_0'' + \frac{1}{2}\Delta t^2 y_0^{(3)} + \frac{3}{16}\Delta t^3 y_0^{(4)} + \frac{1}{32}\Delta t^4 y_0^{(5)} + \frac{1}{128}\Delta t^5 y_0^{(6)}. \end{cases}$$

Therefore hold

$$\text{(2.3)} \qquad \begin{cases} \bar{z}_1 - y'(\Delta t) = \frac{1}{12}\Delta t^3 y_0^{(4)} - \frac{1}{24}\Delta t^4 y_0^{(5)} + O(\Delta t^5) \\ \bar{\bar{z}}_1 - y'(\Delta t) = \frac{1}{48}\Delta t^3 y_0^{(4)} - \frac{1}{96}\Delta t^4 y_0^{(5)} + O(\Delta t^5). \end{cases}$$

If we set

$$U_1 = \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = \frac{1}{3}(4\bar{\bar{U}}_1 - \bar{U}_1),$$

then by (2.2) and (2.3) the terms of $O(\Delta t^3)$ and $O(\Delta t^4)$ vanish, and we have

$$U_1 = \begin{pmatrix} y(\Delta t) \\ y'(\Delta t) \end{pmatrix} + O(\Delta t^5).$$

The vector U_1 is the next approximate value in our extrapolation. The integration scheme consisting of the process

$$U_0 \quad \to \quad \bar{U}_1 = Q_{\Delta t} U_0, \qquad \bar{\bar{U}}_1 = Q_{\Delta t/2}^2 U_0 \quad \to \quad U_1$$

is called the *extrapolated central difference* (ECD) scheme in this paper.

§3. Finite difference expression of the extrapolated central difference scheme

The vectors $\{y_i\}$ determined by the above process do not satisfy the central difference equation (1.4). In this section we seek a difference equation which governs the extrapolated solution. Let $U_{i+1} = (y_{i+1}, z_{i+1})$ be the extrapolated value determined by the starting value $U_i = (y_i, z_i)$:

$$\bar{U}_{i+1} = Q_{\Delta t} U_i, \qquad \bar{\bar{U}}_{i+1} = Q_{\Delta t/2}^2 U_i$$

$$U_{i+1} = \frac{1}{3}(4\bar{\bar{U}}_{i+1} - \bar{U}_{i+1}).$$

Each component is determined as follows.

$$\text{(3.1)} \qquad \begin{pmatrix} y_{i+1} \\ z_{i+1} \end{pmatrix} = \begin{pmatrix} I + \frac{1}{2}\Delta t^2 A + \frac{1}{24}\Delta t^4 A^2 & \Delta t I + \frac{1}{6}\Delta t^3 A \\ \Delta t A + \frac{1}{6}\Delta t^3 A^2 + \frac{1}{96}\Delta t^5 A^3 & I + \frac{1}{2}\Delta t^2 A + \frac{1}{24}\Delta t^4 A^2 \end{pmatrix} \begin{pmatrix} y_i \\ z_i \end{pmatrix}$$

The extrapolated central difference scheme is expressed in the following form.

Theorem 2 *The vectors $\{y_i\}$ determined by (3.1) satisfy the following difference equations.*

$$\frac{y_{i+1}-2y_i+y_{i-1}}{\Delta t^2} = (A+\frac{1}{12}\Delta t^2A^2)y_i - \frac{1}{12\cdot 24}\Delta t^4A^3y_{i-1}. \tag{3.2}$$ □

Since $A=-M^{-1}K$ we have

$$M\frac{(y_{i+1}-2y_i+y_{i-1})}{\Delta t^2}+K(I-\frac{1}{12}\Delta t^2M^{-1}K)y_i-\frac{1}{12\cdot 24}\Delta t^4K(M^{-1}K)^2y_{i-1}=0 \tag{3.3}$$

This is the difference expression of the extrapolated central difference scheme. Note that a little decreasing of stiffness and a very little damping are caused by the extrapolation. See also (4.1) below. Equation (3.3) is called the *extrapolated central difference* equation.

§4. Stability of the extrapolated central difference scheme

It is well known that the central difference scheme (1.4) is stable only under a condition

$$\frac{\Delta t}{h}\le C_1$$

for a certain constant C_1 which depends on the constant C_0 in the inverse inequality and on elastic constants [3]. Is this condition sufficient or relaxed for the extrapolated scheme ? To discuss this problem we introduce a matrix D defined by

$$D=\frac{1}{12}\Delta t^2M^{-1}K$$

and write (3.6) as follows

$$M\frac{(y_{i+1}-2y_i+y_{i-1})}{\Delta t^2}+K(I-D-\frac{1}{2}D^2)y_i+\frac{1}{2}KD^2(y_i-y_{i-1})=0. \tag{4.1}$$

We first derive an energy inequality for this equation. The inner product of the both sides of (4.1) and $y_{i+1}-y_{i-1}$ yields

$$\begin{aligned}&\|\frac{y_{i+1}-y_i}{\Delta t}\|_M^2+(K(I-D-\frac{1}{2}D^2)y_i,y_{i+1})+\frac{1}{2}(KD^2(y_i-y_{i-1}),y_{i+1}-y_{i-1})\\ =\ &\|\frac{y_i-y_{i-1}}{\Delta t}\|_M^2+(K(I-D-\frac{1}{2}D^2)y_{i-1},y_i).\end{aligned}$$

Summing this equation for $i=1,2,\cdots,n$, we have

$$\begin{aligned}&\|\frac{y_{n+1}-y_n}{\Delta t}\|_M^2+(K(I-D-\frac{1}{2}D^2)y_n,y_{n+1})+\frac{1}{2}\sum_{i=1}^{n}(KD^2(y_i-y_{i-1}),y_{i+1}-y_{i-1})\\ =\ &\|\frac{y_1-y_0}{\Delta t}\|_M^2+(K(I-D-\frac{1}{2}D^2)y_0,y_1).\end{aligned} \tag{4.2}$$

To estimate the third term in the left-hand side we set $e_i=y_i-y_{i-1}$ and write

$$\langle e_i,e_j\rangle=(KD^2e_i,e_j),\qquad \|e_i\|^2=\langle e_i,e_j\rangle.$$

Since

$$\begin{aligned}\sum_{i=1}^{n}\langle e_i,e_{i+1}+e_i\rangle &= \sum_{i=1}^{n}\|e_i\|^2+\frac{1}{2}\sum_{i=1}^{n}[\|e_{i+1}+e_i\|^2-\|e_{i+1}\|^2-\|e_i\|^2]\\ &= -\frac{1}{2}\|e_{n+1}\|^2+\frac{1}{2}\sum_{i=1}^{n}\|e_{i+1}+e_i\|^2+\frac{1}{2}\|e_1\|^2,\end{aligned}$$

substituting this into (4.2), we have

$$(4.3)\qquad \|\frac{y_{n+1}-y_n}{\Delta t}\|_M^2 + (K(I-D-\frac{1}{2}D^2)y_n, y_{n+1}) - \frac{1}{4}(KD^2(y_{n+1}-y_n), y_{n+1}-y_n)$$
$$+\frac{1}{4}\sum_{i=1}^{n}(KD^2(y_{i+1}-y_{i-1}), y_{i+1}-y_{i-1})$$
$$= \|\frac{y_1-y_0}{\Delta t}\|_M^2 + (K(I-D-\frac{1}{2}D^2)y_0, y_1) - \frac{1}{4}(KD^2(y_1-y_0), y_1-y_0).$$

Using the identity $ab = \frac{1}{2}(a^2+b^2) - \frac{1}{2}(a-b)^2$, the above equation can be written as follows.

$$(4.4)\qquad \|\frac{y_{n+1}-y_n}{\Delta t}\|_M^2 + \frac{1}{2}[(K(I-D-\frac{1}{2}D^2)y_n, y_n) + (K(I-D-\frac{1}{2}D^2)y_{n+1}, y_{n+1})]$$
$$-\frac{1}{2}(K(I-D)(y_{n+1}-y_n), y_{n+1}-y_n) + \frac{1}{4}\sum_{i=1}^{n}(KD^2(y_{i+1}-y_i), y_{i+1}-y_i)$$
$$= \|\frac{y_1-y_0}{\Delta t}\|_M^2 + \frac{1}{2}[(K(I-D-\frac{1}{2}D^2)y_0, y_0) + (K(I-D-\frac{1}{2}D^2)y_1, y_1)]$$
$$-\frac{1}{2}(K(I-D)(y_1-y_0), y_1-y_0).$$

Therefore, the extrapolated central difference scheme is stable if the following conditions are satisfied

$C(1):\quad K(I-D-\frac{1}{2}D^2)$ is positive definite.

$C(2):\quad \|\frac{z}{\Delta t}\|_M^2 - \frac{1}{2}(K(I-D)z, z)$ is positive for any N – dimensional vector z.

Theorem 3 *Let C_0 be the constant appearing in the inverse inequality (1.2) and set*

$$\alpha = \frac{C_0\Delta t}{h}$$

then the following holds.
(1) The central difference scheme is stable in the sense of energy if

$$(4.5)\qquad \alpha < \sqrt{2}$$

is satisfied.
(2) The condition (4.5) is sufficient for C(1) and C(2). □

The condition

$$(4.6)\qquad \alpha^2 = \left(\frac{C_0\Delta t}{h}\right)^2 \leq 12(\sqrt{3}-1)$$

is sufficient for C(1) and the condition

$$(4.7)\qquad 1 - \frac{1}{2}\alpha^2 + \frac{1}{24}\left(\frac{\Delta t}{C_1}\right)^4 > 0$$

is sufficient for C(2).

As is seen in (4.7) the stability limit is relaxed by the extrapolation. Although it is expected that the admissible time mesh Δt becomes twice that of the central difference scheme, we got negative results in our numerical experience.

To explain this fact, we rewrite (4.7) as follows.

$$1-\frac{1}{2}\alpha^2+\gamma h^4\alpha^4>0, \qquad \gamma=\frac{1}{24}\frac{1}{(C_0C_1)^4}.$$

By solving this inequality we have

$$\alpha^2<\frac{4}{1+\sqrt{1-16\gamma h^4}}.$$

Therefore $\alpha<2$ is the best possible estimate for the extrapolated central difference scheme, so far as the above analysis shows. Compare this result with (4.5).

§5. Modified ECD and Comparisons

The original ECD scheme is written as follows. Let $U_0=(y_0,z_0)$ and $U_1=(y_1,z_1)$ be the initial vector and the next vector to be obtained in one step of the extrapolation, respectively. Three intermediate vectors V_0, V_1 and V_2 are introduced as follows :

$$V_0=\begin{pmatrix}p_0\\q_0\end{pmatrix}=\begin{pmatrix}y_0+\frac{1}{2}\Delta t^2Ay_0+\Delta tz_0\\z_0+\frac{1}{2}\Delta tAy_0+\frac{1}{2}\Delta tAp_0\end{pmatrix}$$

$$V_1=\begin{pmatrix}p_1\\q_1\end{pmatrix}=\begin{pmatrix}y_0+\frac{1}{8}\Delta t^2Ay_0+\frac{1}{2}\Delta tz_0\\z_0+\frac{1}{4}\Delta tAy_0+\frac{1}{4}\Delta tAp_1\end{pmatrix}$$

$$V_2=\begin{pmatrix}p_2\\q_2\end{pmatrix}=\begin{pmatrix}p_1+\frac{1}{8}\Delta t^2Ap_1+\frac{1}{2}\Delta tq_1\\q_1+\frac{1}{4}\Delta tAp_1+\frac{1}{4}\Delta tAp_2\end{pmatrix}.$$

Then U_1 is determined by

$$U_1=\begin{pmatrix}y_1\\z_1\end{pmatrix}=\begin{pmatrix}\frac{1}{3}(4p_2-p_0)\\\frac{1}{3}(4q_2-q_0)\end{pmatrix}.$$

If Ay_0 is stored in the preceding step, the vectors to be calculated in this step are Ap_0, Ap_1 and Ap_2. Here we note that the vectors p_0, p_1 and p_2 and, therefore also y_1 are determined independently of Ap_0 and Ap_2, and that p_0 and p_2 are the approximations to y_1. We therefore introduce the following algorithm.

$$p_0=y_0+\frac{1}{2}\Delta t^2Ay_0+\Delta tz_0$$

$$p_1=y_0+\frac{1}{8}\Delta t^2Ay_0+\frac{1}{2}\Delta tz_0$$

$$q_1=z_0+\frac{1}{4}\Delta tAy_0+\frac{1}{4}\Delta tAp_1$$

$$p_2=p_1+\frac{1}{8}\Delta t^2Ap_1+\frac{1}{2}\Delta tq_1$$

$$y_1=\frac{1}{3}(4p_2-p_0)$$

$$q_0=z_0+\frac{1}{2}\Delta tAy_0+\frac{1}{2}\Delta tAy_1$$

$$q_2 = q_1 + \frac{1}{4}\Delta t A p_1 + \frac{1}{4}\Delta t A y_1$$

$$z_1 = \frac{1}{3}(4q_2 - q_0).$$

In this modified version the calculation of the stiffness matrix appears only in Ap_1 and Ay_1. We will call this scheme the *Modified Extrapolated Central Difference* (MECD) scheme. The stability analysis of MECD can be treated by the same way as for ECD and the similar results are obtained. We note only that the MECD scheme is governed by the following finite difference equation.

$$M\frac{(y_{i+1} - 2y_i + y_{i-1})}{\Delta t^2} + K(I - D)y_i + \frac{1}{6}KD^2(y_i - y_{i-1}) = 0, \tag{5.1}$$

where D is the matrix defined in the preceding section.

The approximation methods compared here are CD, MECD, Midpoint method (MP), Extrapolated Midpoint method (EMP) and 4-th order Runge-Kutta method (RK). As the material we supposed a steel plate in plane stress state. Its configuration and the finite element idealizations are shown in Fig.1 and the mechanical constants are listed in Table 1. The plate is assumed to be fixed at the bottom and the response is observed at the point P. The right-hand side of the differential equations are calculated from the exact solution which is given in advance.

Case (1) : Fig.2 shows the displacement-time diagrams for the case of the coarse mesh, that is, for small systems. In this case the used time mesh $\Delta t = 2$, thought the stability limit for CD is about $\Delta t = 8$. Table 2 shows the maximum relative error and the computing time for each method.

Case (2) : Fig.3 shows the case of the fine mesh. The used time mesh is $\Delta t = 0.5$, where the stability limit for CD is about $\Delta t = 1$. Table 3 shows the maximum relative error and the computing time.

It is observed that in both cases MECD has the same accuracy as RK, but the computing time is about half, and about twice that of CD.

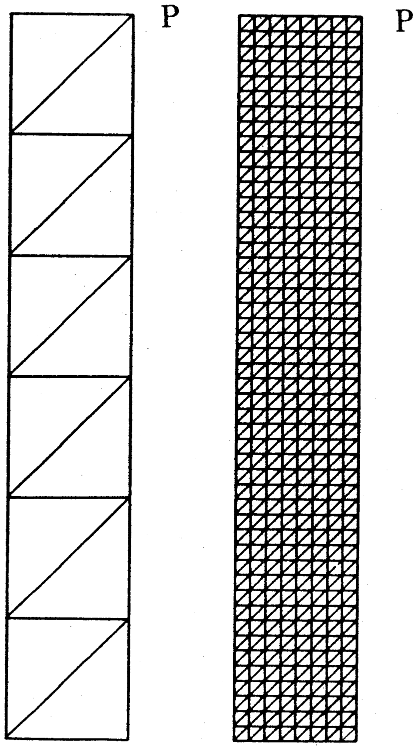

Fig.1 Finite element idealizations

Table 1: The mechanical constans

Young's modulas	206
Poisson's ratio	0.3
Density	7.85×10^3

Table 2: The max.relative errors and comput.time

	The max.relative errors	comput.time
CD	6.05×10^{-0}	0.23 sec
MECD	1.04×10^{-1}	0.53 sec
MP	1.43×10^{-0}	0.45 sec
EMP	5.89×10^{-3}	1.50 sec
RK	1.04×10^{-1}	0.72 sec

Table 3: The max.relative errors and comput.time

	The max.relative errors	comput.time
CD	4.17×10^{-1}	66.33 sec
MECD	4.34×10^{-4}	139.20 sec
MP	1.04×10^{-1}	172.88 sec
EMP	2.77×10^{-5}	512.97 sec
RK	4.39×10^{-4}	271.89 sec

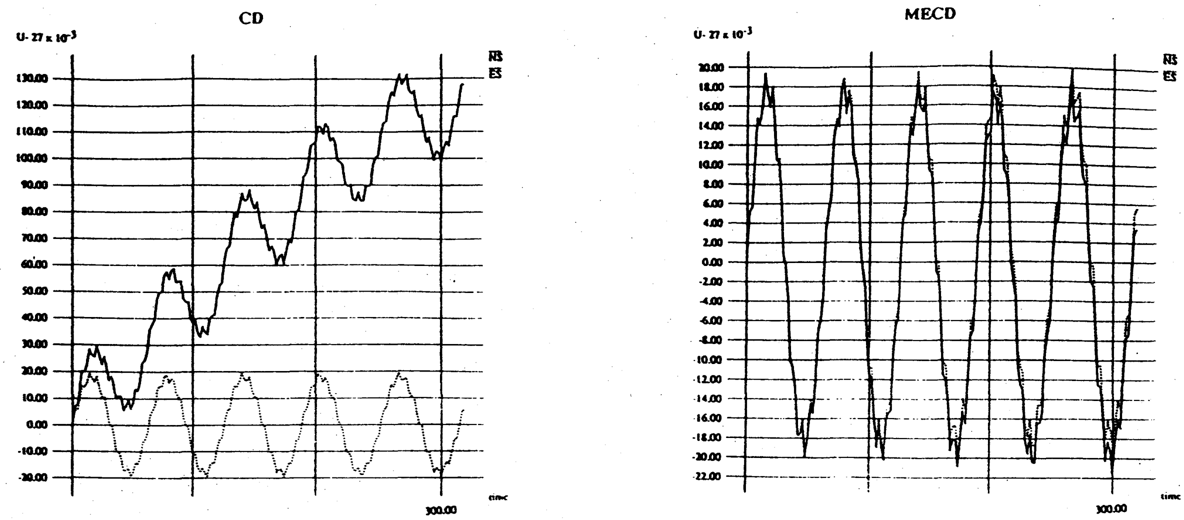

Fig.2 Displacement-time diagrams

NS = Numerical Solutions
ES = Exact Solutions

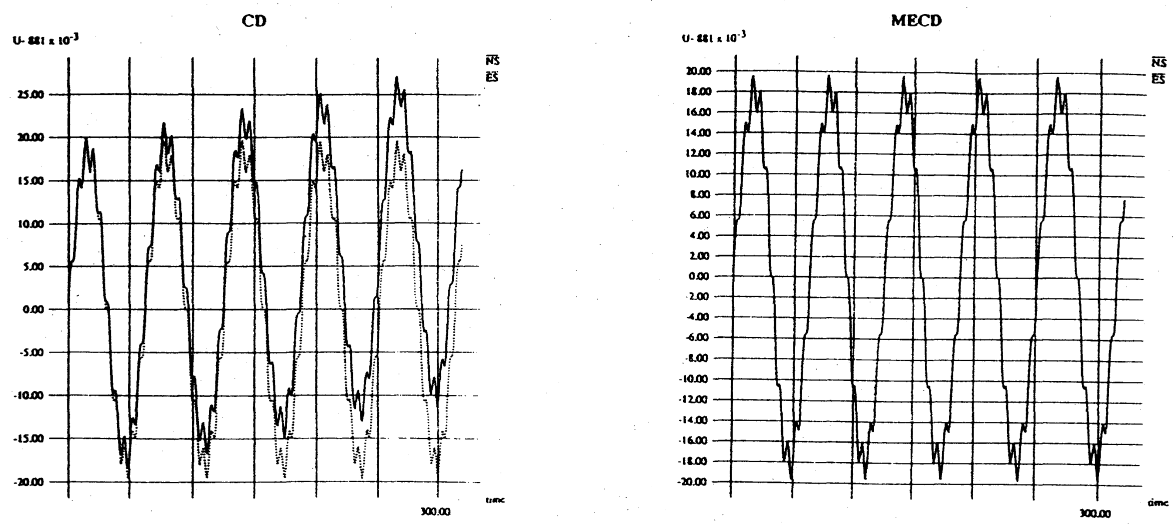

Fig.3 Displacement-time diagrams

References

[1] Adams.A.R *Sobolev Spaces*, Academic Press., 1975.

[2] Ciarlet.P.G *Finite element methods for elliptic problems*, North-Holland Publishing Company., 1978.

[3] Fujii.H *Finite-Element Galerkin Method for Mixed Initial-Boundary Value Problems in Elasticity Theory*, Center for Numerical Analysis The University of Texas at Austin., 1971.

[4] Hsu.T.R. *The Finite Element Method in Thermomechanics*, Allen and Unwin,Inc., 1986.

[5] Joice.D.S. *Survey of Extrapolation Processes in Numerical Analysis*, SIAM Review, Vol.13, No.4 (1971), 435-490.