On conditions for Teichmüller mappings of the unit disk

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We consider quasiconformal self-mappings $f = f^\kappa$ of the unit disk $E = \{|z| < 1\}$. The superscript κ denotes the complex dilatation of the mapping f. By continuation, f induces a homeomorphism of the boundary ∂E onto itself. Let Q_f be the class of all quasiconformal self-mappings of E with the same boundary values as f, namely, $g \in Q_f$ if and only if $g(e^{i\theta}) = f(e^{i\theta})$. It is well known and can be proved by a normal family argument that there exists an extremal quasiconformal mapping f_0 in Q_f, i.e., a mapping with the smallest maximal dilatation K_0 in the class Q_f:

$$K_0 = \inf_{g \subseteq Q_f} K[g],$$

where $K[f] = K$ denotes the maximal dilatation of f. Because of the relation $K = (1 + k)/(1 - k)$, $k = \|\kappa\|_\infty$, this is equivalent to the condition

$$k_0 = \inf_{g \subseteq Q_f} k[g].$$

1. Preliminaries.

For $0 < \theta < \pi$, the open sector with vertex angle 2θ about the positive real axis is denoted by

$$S(\theta) = \{re^{it} \mid 0 < r < \infty, |t| < \theta\}.$$

For $0 \leq \theta_1, \theta_2 \leq 2\pi$, similary, we denote closed and open angular sectors by

$$S^k[\theta_1, \theta_2] = \{re^{it} \mid 0 < r \leq k, \theta_1 \leq t \leq \theta_2\},$$

and

$$S^k(\theta_1, \theta_2) = \{re^{it} \mid 0 < r \leq k, \theta_1 < t < \theta_2\}$$

respectively.

We set $\operatorname{sgn} z = \frac{z}{|z|}$ for $z \neq 0$, and $\operatorname{sgn} 0 \equiv 0$.

Let B be the set of all functions f which are holomorphic on E and satisfy

$$\|f\|_1 = \iint_E |f| \, dx \, dy < \infty,$$

and $B_1 = \{f \in B : \|f\|_1 = 1\}$, the boundary of the unit ball in B.

Let $C(B)$ denote the infimun of the set of all $C \in (0, \infty]$ such that

$$\iint_E |f(z)| \, dx \, dy \leq C \iint_E |\operatorname{Re} f(z)| \, dx \, dy,$$

where $f \in B$ and $\operatorname{Im} f(0) = 0$.

The extremal mapping f_0 need not be uniquely determined and therefore the extremal complex dilatation κ_0 need not be unique either. Uniqueness theorems are known for Teichmüller mappings, i.e. mappings with a complex dilatation of the form

$$(1) \qquad \kappa(z) = k\overline{\varphi(z)}/|\varphi(z)|,$$

where $\varphi(z)$ is homomorphic in E. For instance, for a holomorphic φ with finite norm $\|\varphi\| = \iint_E |\varphi(z)| \, dx \, dy < \infty$, f^κ is uniquely extremal in Q_f. For a φ with infinite norm, f^κ need not be extremal, and even if it is , it need not be unique.

We will consider two kinds of problems

A) If $f(z)$ is an extremal quasiconformal mapping, under what condition does the mapping $f(z)$ happen to be a Teichmüller mapping?

B) If $f(z)$ is a Teichmüller mapping, in what case is the mapping an extremal mapping?

2. Conditions for an extremal mapping to be a Teichmüller mapping.

It is evidently of interest to characterize the complex dilatation κ_0 of an arbitrary extremal quasiconformal mapping. The fundamental characterization of extremal dilatations is due to R. Hamilton [1], E. Reich, and K. Strebel [2].

THEOREM 1. *A dilatation κ is extremal if and only if one of the following statements is true:*

1) There exist $f \in B_1$ and $k \in [0,1)$ such that $\kappa(z) = k \operatorname{sgn} \overline{f(z)}$ for almost all $z \in E$.

2) There is a sequence $\{f_n\}_1^\infty$, of elements of B_1, converging to zero uniformly on compact subsets of E, such that

$$\lim_{n \to \infty} \left| \iint_E f_n(z)\kappa(z) \, dx \, dy \right| = \|\kappa\|_\infty.$$

For an explicitly given $\kappa(z)$, it is usually difficult to check the above condition 2). For this purpose, some special classes of dilatations have been studied. It is hoped to derive some characterization of extremal dilatation which is more explicit than that in Theorem 1. A. Harrington, E. Reich, K.Strebel, M. Ortel (for example, see [3]) and others had obtained such characterizations of the extremals within special classes of dilatations. Recently M. Ortel and W. Smith, in [4], investigated the argument of an extremal dilatation and proved the following

THEOREM 2. *Suppose that $\kappa(z)$ is a bounded, Lebesgue measurable function on E, such that $\|\kappa(z)\|_\infty < 1$, $\frac{\pi}{2} < \theta < \frac{\pi}{2} + \arctan \frac{1}{2C(B)}$, and $\operatorname{sgn} \kappa(z) \in S(\theta) \cup \{0\}$ for almost all $z \in E$. Then $\kappa(z)$ is an extremal dilatation if and only if there exist $f(z) \in B_1$ and $k \in [0,1)$ such that*

$$\kappa(z) = k \operatorname{sgn} \overline{f(z)}$$

for almost all $z \in E$.

Our first work concerns with the image domain of an extremal dilatation. By setting up some modified restrictions which work out as a necessary and sufficient condition for extremal dilatation, it turns out that the extremal quasiconformal mapping is a Teichmüller mapping. We obtain that, in [5],

THEOREM 3. *Suppose that $\kappa(z)$ is a bounded Lebesgue measurable function on E. The norm $\|\kappa(z)\|_\infty = k < 1$ and $\frac{\pi}{2} < \theta < \frac{\pi}{2} + \arcsin(\frac{1}{2C(B)-1})$. Also suppose that there exists $k' < k$ such that $\kappa(z) \in S^k[-\theta, \theta] \cup S^{k'}(\theta, 2\pi - \theta) \cup \{0\}$ for almost all*

$z \in E$. Then $\kappa(z)$ is an extremal dilatation if and only if there exists $f(z) \in B_1$ such that

$$\kappa(z) = k \, \text{sgn} \, \overline{f(z)}$$

for almost all $z \in E$.

Even in the special case $k' = 0$ in Theorem 3, this theorem improves the above Theorem 2 obtained by M.Ortel and W. Smith [4].

3. Conditions for a Teichmüller mapping to be extremal.

On the other hand, if $f(z)$ is a Teichmüller mapping with complex dilatation

$$(1) \qquad \kappa(z) = k \frac{\overline{\varphi(z)}}{|\varphi(z)|}, \qquad z \in E, \, 0 < k < 1,$$

where $\varphi(z)$ is holomorphic in E, and $\|\varphi(z)\| = \infty$, when is the mapping $f(z)$ extremal for its boundary values? Furthermore, if $f(z)$ is an extremal mapping, does $\{\varphi(R_n z)\}$, where $0 < R_n < 1$ and $\lim_{n \to \infty} R_n = 1$, constitute a Hamilton sequence? To this question, E. Reich showed in [8] that

THEOREM 4. Suppose that $\varphi(z)$ is holomorphic in a neighborhood of $\overline{E}$ except for a finite number of poles on ∂E. Then

$$\lim_{R \to 1} \frac{\left| \iint_E \frac{\overline{\varphi(z)}}{|\varphi(z)|} \varphi(Rz) \, dx \, dy \right|}{\|\varphi(z)\|} = 1,$$

if and only if $\varphi(z)$ has poles of at most order 2 on ∂E.

From the work of Sethares [7], Theorem 4 is equivalent to the following

THEOREM 5. Suppose that $\varphi(z)$ is holomorphic in a neighborhood of $\overline{E}$ except for a finite number of poles on ∂E. Then, $\{\varphi(R_n z)\}$ is a Hamilton sequence if and only if the Teichmüller mapping with dilatation (1) is uniquely extremal.

In the work of Sethares [7], he obtained extremality and uniqueness theorems for Teichmüller mappings with $\|\varphi(z)\| = \infty$ by means of certain assumption on the

growth of $\sup_{|z|=r} |\varphi(z)|$. However, in [2], Reich and Strebel obtained the following extremality theorem under a growth assumption on $I_1(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} |\varphi(re^{i\theta})| \, d\theta$.

THEOREM 6. *Suppose that φ is holomorphic in E, and*

$$\frac{1}{2\pi} \int_0^{2\pi} |\varphi(re^{i\theta})| \, d\theta = O(\frac{1}{1-r}), \quad r \to 1.$$

Then the Teichmüller mapping with dilatation (1) is extremal for its boundary values.

Later, in [9], W. K. Hayman and E. Reich showed that

THEOREM 7. *Suppose that $\varphi(z)$ is holomorphic in E, and*

$$\int_0^{2\pi} |\varphi(re^{i\theta})| \, d\theta \le \frac{1}{1-r}, \quad 0 \le r < 1.$$

Then the Teichmüller mapping (1) is uniquely extremal and $\{\varphi(R_n z)\}$ is a Hamilton sequence.

We prove here a extremality theorem under a growth assumption on

$$A(r, \varphi) = \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} |\varphi(re^{i\theta})| r \, dr \, d\theta.$$

We know, from Hardy's convexity theorem, that $I_1(r, \varphi)$ is a nondeceasing function of r, and

$$A(r, \varphi) \le I_1(r, \varphi) \le \sup_{|z|=r} |\varphi(z)|.$$

Our results can be stated as follows

THEOREM 8. *Suppose that $\varphi(z)$ is holomorphic in E,*

$$\lim_{r \to 1} \frac{\log \frac{1}{1-r}}{A(r, \varphi)} = 0,$$

and

$$I_1(r, \varphi) \le O(\frac{1}{1-r} \log \frac{1}{1-r}), \quad r \to 1.$$

Then the Teichmüller mapping with dilatation (1) is extremal for its boundary values, and there exists a sequence of numbers $\{\tilde{R}_n\}$ such that $0 < \tilde{R}_n < 1$, $\lim_{n\to\infty} \tilde{R}_n = 1$, and $\{\varphi(\tilde{R}_n z)\}$ is a Hamilton sequence.

The proof will be given in [6]. As an application of Theorem 8, we obtain the following two corollaries

COROLLARY 1. *Suppose that $\varphi(z)$ is holomorphic in E,*

$$I_1(r, \varphi) = O(\frac{1}{1-r} \log \frac{1}{1-r}), \quad r \to 1,$$

and

$$\varlimsup_{r\to 1}(1-r)A(r, \varphi) \neq 0.$$

Then the Teichmüller mapping with dilatation (1) is extremal and there exists a sequence of numbers $\{\tilde{R}_n\}$ such that $0 < \tilde{R}_n < 1$, $\lim_{n\to\infty} \tilde{R}_n = 1$, and $\{\varphi(\tilde{R}_n z)\}$ is a Hamilton sequence.

COROLLARY 2. *Suppose that $\varphi(z)$ is holomorphic in E,*

$$I_1(r, \varphi) = O(\frac{1}{1-r} \log \frac{1}{1-r}), \quad r \to 1,$$

and

$$\lim_{r\to 1}(1-r)I_1(r, \varphi) = \infty.$$

Then the Teichmüller mapping with dilatation (1) is extremal and $\{\varphi(R_n z)\}$ is a Hamilton sequence for any $\{R_n\}$ with $0 < R_n < 1$ and $\lim_{n\to\infty} R_n = 1$.

Recall that, in the case of $\lim_{r\to 1}(1-r)I_1(r, \varphi) = a \neq \infty$, the result had been proved in [2]. Also the following example shows that Theorem 8 is really a generalization of Theorem 6.

Set

$$\varphi(z) = \frac{\log \frac{1}{1-z}}{(1-z)^2},$$

then the Teichmüller mapping $f(z)$ with dilatation $k\overline{\varphi(z)}/|\varphi(z)|$, $0 < k < 1$, is extremal for its boundary values.

In this case, we have,

$$\frac{C_1}{1-r}\log\frac{1}{1-r} \leq I_1(r,\varphi) \leq \frac{C_2}{1-r}\log\frac{1}{1-r}, \quad r \to 1,$$

with suitable $C_1, C_2 > 0$.

However, we obtain that there exists a constant $C_3 > 0$ such that

$$A(r,\varphi) \geq C_3(\log\frac{1}{1-r})^2, \quad r \to 1,$$

and hence

$$\lim_{r \to 1}\frac{\log\frac{1}{1-r}}{A(r,\varphi)} = 0.$$

By Theorem 8, the Teichmüller mapping $f(z)$ with dilatation $k\overline{\varphi(z)}/|\varphi(z)|$, $0 < k < 1$, is extremal for its boundary values. But $\varphi(z)$ does not satisfy the conditions in Theorem 6.

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