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Ideal triangulations of noncompact hyperbolic 3–manifolds

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1 Introduction

Epstein and Penner [1] showed that every noncompact complete hyperbolic manifold of finite volume admits canonical ideal polyhedral decomposition. Here "ideal polyhedral decomposition" means that all the edges are geodesic and all the faces are totally geodesic and all the vertices correspond to the cusps of the manifold. In this article we are only concerned with hyperbolic manifolds with the same conditions as above, so we can always assume that some cusps exist and this definition of "ideal polyhedral decomposition" makes sense. We have asked whether the above "ideal polyhedral decomposition" can be replaced with "ideal triangulation" — i.e. tetrahedral decomposition — though it may not be canonical anymore. The arguments of [1] work in any dimension but in the following we are only concerned with three dimensional case. (In dimension less than three it is easy. In dimension greater than three we can not say anything.)

If we can get a triangulation of a given hyperbolic 3–manifold, we can calculate the explicit hyperbolic structure and especially the deformations of the hyperbolic structures of this manifold easier than working on only with polyhedral decomposition. Such deformations are investigated only in the very simplest cases and one of the difficulties to proceed is the amount of the calculations. So we hope that our way of getting the simple expression of the hyperbolic manifolds will be helpful to do such calculations.

Our second motivation comes from Weeks's works. He has developed a mathematical computer software called "Snappea". Using this nice software he made the list of all hyperbolic 3–manifolds which can be constructed by at most 7 ideal tetrahedra with the faces glued with each other. If there exist some counter examples of our question, his way can not list all the noncompact complete hyperbolic 3-manifolds with finite volume from the simplest ones. But experiments showed that decomposition by tetrahedra is always possible. This experience is another motivation of this work.

We could not get the complete answer to the question. But some partial answer is possible. Here is our main result.

Theorem 1 *Let M be a hyperbolic 3–manifold. If the canonical decomposition of M in the sense of [1] consists of one polyhedron, we can subdivide this decomposition to get the ideal triangulation of this manifold. If it consists of two polyhedra and one of them has all the faces glued only to the other polyhedron (i.e. without self gluing), then the same conclusion holds.*

Remark 1 Though one of our motivation is [1], the triangulations that we want do not have to be the subdivisions of Epstein and Penner's. But our argument depends their work and we don't know any other way without passing their decompositions to the triangulations. There are still possibilities that 1. our question has counter examples. or 2. our question does not have any counter examples but it is not possible to get the triangulation from the subdivision of the Epstein and Penner's.
Remark 2 Our primary concern is to investigate the geometry of the hyperbolic manifolds. But our triangulations which is not canonical could break the symmetry the original polyhedral decomposition had. We should be careful on this setback.

2 Sketch of the proof

As we have mentioned in remark 1 our starting point is the canonical polyhedral decomposition of Epstein and Penner's. In the following we show that in some simple cases it is possible to subdivide the polyhedra to get tetrahedra. General idea is that for given convex polyhedra with the identifications of their faces specified (and with the conditions in the theorem satisfied), we will try to show that we can subdivide them into tetradehra in a way that each face of the tetrahedra is glued to only one face of another tetrahedron i.e. it gives a triangulation — we say that it is "compatible" with the original identification.

We first remark that our argument in this section is purely combinatorial and does not use any geometry. For now the author don't know whether geometry is really needed to get the complete answer to our question.

2.1

Let P be a (ideal) polyhedron . Note that the triangulation (subdivision into tetrahedra) of P is always possible if we forget the gluing.

We'll do this by specifying one vertex v of P and regard P as a cone with top vertex v and regard the faces adjacent to v as the slopes of the cone and the other faces of P as the bottom faces of the cone.

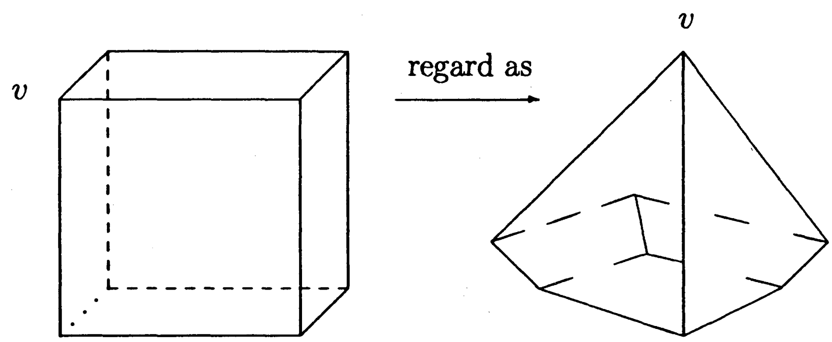

Figure 1

Then triangulation of the all "bottom" faces gives us the collection of the tetrahedra each of which having the triangles as the bottom face and having top vertex v in common.

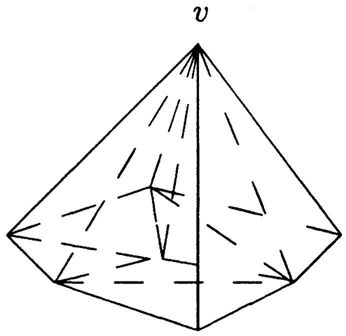

Figure 2

We can subdivide the bottom faces arbitrary but the triangulation of the faces regarded as slopes can not be changed — all the diagonal edges should be adjacent to v.

Given polyhedra with identifications of the faces specified (which come from Epstein and Penner's decomposition), to solve our problem, we try to find top vertices for each polyhedra whose tetrahedral decomposition is compatible with respect to the specified identifications.

Picking up a vertices for every polydehra as the top vertices means that every faces including these vertices (slope) will be divided by the lines adjacent to this vertex in the triangulation. But how the other faces (bottom) will be divided can be chosen whatever we like at this point to make this decomposition compatible.

If there exist slope faces with respect to the chosen top vertices having subdivision conflicting with each other, this set of vertices is not suitable to get the compatible one.

Our next step is to show the existence of "good" set of vertices.

2.2

If the decomposition consists of one polyhedron P, it is fairly easy to show the existence of good vertex.

In this case top vertex that we have to choose is only one and triangulation associated with this vertex could be incompatible if some of the faces including this vertex are identified with each other. To make this situation happen these two faces must be adjacent. In other words each identification of the faces can produce at most two "bad" vertices (two: when two faces are connected by an edge, one: when connected by a vertex) .

Now we have some calculation. Let V, E, F denote the set of all vertices, edges, and faces of the polehedron P. We write the number of good and bad vertices by V_G and V_B respectively. Let F_n be the number of n–gonal faces of the polydegron.

The above argument showed that:

$$\begin{aligned} V_G &= \#V - V_B \\ &\geq \#V - \sum_{n\geq 4} F_n \end{aligned}$$

(Note that pair of triangle faces does not cause any vertex becoming bad because they don't have to be subdivided.)

By the Euler's formula:

$$\begin{aligned} \#V &= 2 + \#E - \#F \\ &= 2 + \frac{1}{2}\sum_{n\geq 3} nF_n - \sum_{n\geq 3} F_n \\ &= 2 + \sum_{n\geq 3}(\frac{n}{2} - 1)F_n \\ &\geq 2 + \sum_{n\geq 4}(\frac{n}{2} - 1)F_n \\ &\geq \sum_{n\geq 4}(\frac{n}{2} - 1)F_n \\ &\geq \sum_{n\geq 4} F_n \end{aligned}$$

If follows that

$$V_G > 0$$

We can say that there exist at least one good vertex and P has is subdivision compatible with respect to its identifications.

2.3

The case when we have two polyhedra P_1, P_2 is not so easy. Let us assume that the faces of the polyhedra are glued only to the other polyhedron's faces.

(This condition is stronger than we have mentioned in the theorem. But we restrict our attentions to this condition for the sake of simplicity.) Note that using the Euler's formula and the above conditions, the number of vertices, edges, faces of P_i's are the same.

As in the above argument let us call $(v_1, v_2) \in V_1 \times V_2$, where V_i is the set of all vertices of P_i, "good pair" if the associated triangulation is compatible. "bad pair" otherwise.

Again we count the number of bad pairs by the gluing of the faces.

Let us think about what happens when two pentagons are glued together.

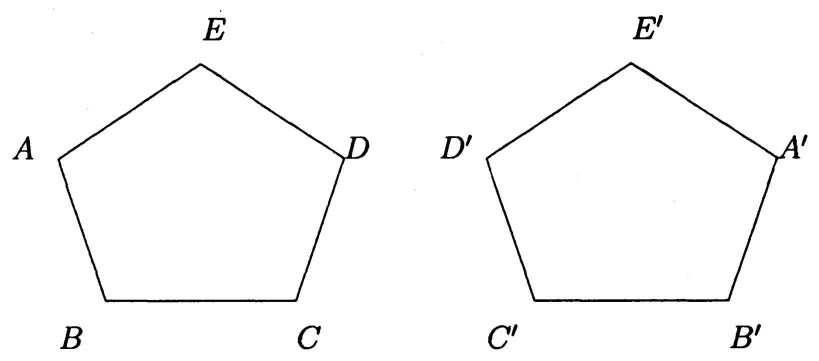

Figure 3

Look at the figure 3. Suppose that A, B, C, D, E are identified with A', B', C', D', E'. Then the bad pairs are:

$$\begin{array}{ccccc}
 & (A,B') & (A,C') & (A,D') & (A,E') \\
(B,A') & & (B,C') & (B,D') & (B,E') \\
(C,A') & (C,B') & & (C,D') & (C,E') \\
(D,A') & (D,B') & (D,C') & & (D,E') \\
(E,A') & (E,B') & (E,C') & (E,D') &
\end{array}$$

and the number is $20 = 5 \times (5-1)$. The same arguments show that:

triangle no bad pair occurs by the gluing of triangles

square $8(= 4 \times 2)$ bad pairs (a little bit different from $n \geq 5$)

n-gon n(n-1) bad pairs ($n \geq 5$)

Let $D : F \to \{0, 1, \ldots\}$ where F is the set of faces of the polehedron P_1 be defined as follows:

$$D(f) = \begin{cases} 0 & \text{if f is triangle} \\ 8 & \text{if f is square} \\ n(n-1) & \text{if f is n-gon}(n \geq 5) \end{cases}$$

Then we have to show that

$$\#\{\text{all pairs}\} \;=\; \#V^2 > \sum_{v \in V} \sum_{f \ni v} D(f).$$

This inequation follows from a complecated computations and calculations so we omit here. Difficulties come from irregular behaviors of D when the face is triangle or square. Also if the polyhedra are big enough it becomes easier to show the existence of a good pair.

Remark The above inequation can be sharpened to make the condition of the theorem weaker. But to prove (or disprove?) our first question, it seems that we need other methods to work on.

References

[1] Epstein,D.B.A. and Penner,R., Euclidean decomposition of noncompact hyperbolic manifolds, J.Differential Geom.,**27**(1988),67–80.