数理解析研究所講究録
第 890 巻 1994 年 19-30

Inverse source problems in Poisson's equations

SABUROU SAITOH (斎 藤 三 郎)

Abstract – In the Poisson's equations

$$\Delta u = -\rho(\mathbf{r}) \quad \text{on} \quad \mathbb{R}^1, \mathbb{R}^2 \quad \text{and} \quad \mathbb{R}^3$$

for $L_2(d\mathbf{r})$ functions ρ, the natural and fundamental inverse formulas representing ρ in terms of the values of u in the outsides of supports of ρ are established.

1. INTRODUCTION

We shall consider the Poisson's equations

$$\Delta u = -\rho(\mathbf{r}) \quad \text{on} \quad \mathbb{R}^1, \mathbb{R}^2 \quad \text{and} \quad \mathbb{R}^3 \tag{1.1}$$

for real-valued $L_2(d\mathbf{r})$ source functions ρ whose supports are contained in a sphere $r < a$ or in the outside of the sphere; r denotes the usual distance $r = |\mathbf{r}|$ from the origin. By using the general method ([3] and [4]) for integral transforms using the theory of reproducing kernels, we first give the characterizations and natural representations of the potentials u on the outsides of supports of ρ. As an application, we shall give surprisingly simple expressions of ρ^* in terms of u of the outsides of supports of ρ, which have the minimum $L_2(d\mathbf{r})$ norms among the source functions ρ satisfying (1.1) on the outsides of supports of ρ. These representations give practical applications to determine the source functions ρ^* by the potentials u. These inverse problems were proposed by Laplace 200 years ago and many mathematicians have attacked to these problems. In this paper, we shall give the reasonable solutions for these problems in the framework of $L_2(d\mathbf{r})$ spaces. See Foreword and References of Isakov [1]. For a basic and general reference for the inverse source problems, see [1].

2. CASE WITH COMPACT SUPPORT ON $\mathbb{R}^3$

We assume that the supports of ρ are contained in the sphere $r < a$ with radius a. So, for $\mathbf{r} \in \mathbb{R}^3$ and $|\mathbf{r}| = r$, we shall examine the integral representation of the solutions u of the Poisson equation (1.1)

Department of Mathematics, Faculty of Engineering, Gunma University, Kiryu 376, JAPAN
E-mail address : ssaitoh@eg.gunma-u.ac.jp

$$u(\mathbf{r}') = \frac{1}{4\pi}\int_{r<a} \frac{1}{|\mathbf{r}' - \mathbf{r}|}\rho(\mathbf{r})d\mathbf{r} \tag{2.1}$$

on $\mathbb{R}^3$ for the source functions ρ satisfying

$$\int_{r<a} \rho(\mathbf{r})^2 d\mathbf{r} < \infty. \tag{2.2}$$

In order to determine the characteristic property of the potentials u on $r > a$ using the general method ([3] and [4]) for integral transforms, we consider the kernel form

$$K_{i,a}(\mathbf{r}', \mathbf{r}'') = \frac{1}{4\pi}\int_{r<a} \frac{1}{|\mathbf{r}' - \mathbf{r}||\mathbf{r}'' - \mathbf{r}|}d\mathbf{r}$$

for $r', r'' > a$. In order to calculate $K_{i,a}(\mathbf{r}', \mathbf{r}'')$ we shall use the expansion

$$\frac{1}{|\mathbf{r}' - \mathbf{r}|} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{\varepsilon_m (n-m)!}{(n+m)!} P_n^m(\cos\theta') P_n^m(\cos\theta)\cos m(\varphi' - \varphi)\frac{r^n}{r'^{n+1}}$$
$$\text{for} \quad r' > r \tag{2.3}$$

in terms of spherical coordinates (r, θ, φ) and (r', θ', φ') (cf. [2, p. 1274]). Here, $\varepsilon_m = 2 - \delta_{mo}$ is the Neumann factor. By using the two orthogonality

$$\int_0^{2\pi} \cos m(\varphi' - \varphi)\cos m'(\varphi'' - \varphi)d\varphi = 2\pi\delta_{mm'}(\varepsilon_m)^{-1}\cos m(\varphi' - \varphi'') \tag{2.4}$$

and

$$\begin{aligned} &\int_0^{\pi} P_n^m(\cos\theta)P_{n'}^m(\cos\theta)\sin\theta d\theta \\ &= \int_{-1}^{1} P_n^m(x)P_{n'}^m(x)dx \\ &= \delta_{nn'}\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}, \end{aligned} \tag{2.5}$$

we have the expansion

$$\begin{aligned} K_{i,a}(\mathbf{r}', \mathbf{r}'') = &\sum_{n=0}^{\infty} \frac{a^{2n+3}}{(2n+1)(2n+3)}\frac{1}{r'^{n+1}}\frac{1}{r''^{n+1}} \\ &\times \sum_{m=0}^{n} \frac{\varepsilon_m(n-m)!}{(n+m)!}P_n^m(\cos\theta')P_n^m(\cos\theta'') \\ &\times (\cos m\varphi' \cos m\varphi'' + \sin m\varphi' \sin m\varphi'') \\ &\quad \text{for} \quad r', r'' > a. \end{aligned} \tag{2.6}$$

This series converges absolutely on $r', r'' > a$. The kernel $K_{i,a}(\mathbf{r}', \mathbf{r}'')$ is a positive matrix on $r > a$ and so , there exists a uniquely determined Hilbert space $H_{K_{i,a}}$ admitting the reproducing kernel $K_{i,a}(\mathbf{r}', \mathbf{r}'')$. Furthermore, the images $u(\mathbf{r}')$ of (2.1) belong just to the Hilbert space $H_{K_{i,a}}$. The expansion (2.6) implies that the images $u(\mathbf{r}')$ are expressible in the form

$$u(\mathbf{r}') = \sum_{n=0}^{\infty} \frac{a^{2n+3}}{(2n+1)(2n+3)} \frac{1}{r'^{n+1}} \times \sum_{m=0}^{n} \frac{\varepsilon_m (n-m)!}{(n+m)!} P_n^m(\cos\theta')(A_n^m \cos m\varphi' + B_n^m \sin m\varphi') \tag{2.7}$$

for some constants $\{A_n^m, B_n^m\}_{n,m=0}^{\infty}$ satisfying

$$\sum_{n=0}^{\infty} \frac{a^{2n+3}}{(2n+1)(2n+3)} \sum_{m=0}^{n} \frac{\varepsilon_m (n-m)!}{(n+m)!} \{(A_n^m)^2 + (B_n^m)^2\} < \infty. \tag{2.8}$$

Conversely, any $u(\mathbf{r}')$ defined by (2.7) with (2.8) belongs to $H_{K_{i,a}}$. Since the family

$$\{P_n^m(\cos\theta)\cos m\varphi, P_n^m(\cos\theta)\sin m\varphi\}_{n,m=0}^{\infty}$$

is complete in the Hilbert space composing of the functions $f(\theta, \varphi)$ with finite norms

$$\left\{ \int_0^{\pi} \int_0^{2\pi} f(\theta,\varphi)^2 \sin\theta d\theta d\varphi \right\}^{\frac{1}{2}} < \infty,$$

we have the representation of the norm $\|u\|_{H_{K_{i,a}}}$ in the form

$$\|u\|_{H_{K_{i,a}}}^2 = \sum_{n=0}^{\infty} \frac{a^{2n+3}}{(2n+1)(2n+3)} \sum_{m=0}^{n} \frac{\varepsilon_m (n-m)!}{(n+m)!} \{(A_n^m)^2 + (B_n^m)^2\}. \tag{2.9}$$

Furthermore, we have the isometrical identity

$$\|u\|_{H_{K_{i,a}}}^2 = \min \int_{r<a} \varphi(\mathbf{r})^2 d\mathbf{r} = \int_{r<a} \varphi^*(\mathbf{r})^2 d\mathbf{r}. \tag{2.10}$$

Here, the minimum is taken over all $\rho(\mathbf{r})$ satisfying (2.1) and (2.2) for $r' > a$ and φ^* is the uniquely determined function with the minimum norm ([3] and [4]).

In (2.7), by using the orthogonality

$$\int_0^{\pi}\int_0^{2\pi} P_n^m(\cos\theta)\cos m\varphi P_{n'}^{m'}(\cos\theta)\left\{\begin{matrix}\cos m'\varphi\\ \text{or}\\ \sin m'\varphi\end{matrix}\right\}\sin\theta d\theta d\varphi$$
$$= \delta_{mm'}\delta_{nn'}\frac{4\pi(\varepsilon_m)^{-1}}{2n+1}\frac{(n+m)!}{(n-m)!},$$

we have the expressions, for any fixed $b > a$.

$$A_n^m = \frac{(2n+1)^2(2n+3)b^{n+1}}{4\pi a^{2n+3}}\int_0^{\pi}\int_0^{2\pi} u(b,\theta,\varphi)$$
$$\times P_n^m(\cos\theta)\cos m\varphi\sin\theta d\theta d\varphi \tag{2.11}$$

and

$$B_n^m = \frac{(2n+1)^2(2n+3)b^{n+1}}{4\pi a^{2n+3}}\int_0^{\pi}\int_0^{2\pi} u(b,\theta,\varphi)$$
$$\times P_n^m(\cos\theta)\sin m\varphi\sin\theta d\theta d\varphi. \tag{2.12}$$

These expressions with (2.7) imply that for any point $\mathbf{r}'(r' > a)$, the potentials $u(\mathbf{r}')$ are expressible in terms of

$$u(b,\theta,\varphi) \quad \text{for any fixed} \quad b(b > a). \tag{2.13}$$

We shall derive the inverse formula representing ρ^* in terms of (2.13).
Using the reproducing property of $K_{i,a}(\mathbf{r},\mathbf{r}')$ in $H_{K_{i,a}}$, we have

$$u(\mathbf{r}') = (u(\mathbf{r}), K_{i,a}(\mathbf{r},\mathbf{r}'))_{H_{K_{i,a}}}$$
$$= \left(u(\mathbf{r}), \frac{1}{4\pi}\int_{r_1<a}\frac{1}{|\mathbf{r}-\mathbf{r}_1||\mathbf{r}'-\mathbf{r}_1|}d\mathbf{r}_1\right)_{H_{K_{i,a}}}$$
$$= \frac{1}{4\pi}\int_{r_1<a}\frac{d\mathbf{r}_1}{|\mathbf{r}'-\mathbf{r}_1|}(u(\mathbf{r}), \frac{1}{|\mathbf{r}-\mathbf{r}_1|})_{H_{K_{i,a}}}$$
$$= \frac{1}{4\pi}\int_{r_1<a}\frac{d\mathbf{r}_1}{|\mathbf{r}'-\mathbf{r}_1|}\rho^*(\mathbf{r}_1) \tag{2.14}$$

and so, we have

$$\rho^*(\mathbf{r}_1) = (u(\mathbf{r}), \frac{1}{|\mathbf{r}-\mathbf{r}_1|})_{H_{K_{i,a}}}. \tag{2.15}$$

Here, note that $\frac{1}{|\mathbf{r}-\mathbf{r}_1|}$ belongs to the Hilbert space $H_{K_{i,a}}$.

Indeed, from (2.7) and (2.3) the corresponding coefficients $\widetilde{A_n^m}$ and $\widetilde{B_n^m}$ of $\frac{1}{|\mathbf{r}-\mathbf{r}_1|}$ in the representation (2.7) are

$$\widetilde{A_n^m} = \frac{(2n+1)(2n+3)}{a^{2n+3}} P_n^m(\cos\theta_1) r_1^n \cos m\varphi_1$$

and

$$\widetilde{B_n^m} = \frac{(2n+1)(2n+3)}{a^{2n+3}} P_n^m(\cos\theta_1) r_1^n \sin m\varphi_1.$$

Then,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{a^{2n+3}}{(2n+1)(2n+3)} \sum_{m=0}^{n} \frac{\varepsilon_m (n-m)!}{(n+m)!} \{(\widetilde{A_n^m})^2 + (\widetilde{B_n^m})^2\} \\
&= \sum_{n=0}^{\infty} \frac{(2n+1)(2n+3)}{a^{2n+3}} \sum_{m=0}^{n} \frac{\varepsilon_m (n-m)!}{(n+m)!} P_n^m(\cos\theta_1)^2 r_1^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(2n+1)(2n+3) r_1^{2n}}{a^{2n+3}} P_n(1) \\
&= \sum_{n=0}^{\infty} \frac{(2n+1)(2n+3) r_1^{2n}}{a^{2n+3}} < \infty \quad \text{for} \quad r_1 < a. \qquad (2.16)
\end{aligned}$$

See [2, p. 1274].

Hence, the formal arguments in (2.14) and (2.15) are justified. See also the following (2.17) and [4] for these arguments. We thus have

Theorem 2.1. The source functions ρ^* in the sense of (2.10) in (1.1) are expressible in terms of (2.13) in the form

$$\begin{aligned}
\rho^*(\mathbf{r}_1) &= \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{(2n+1)^2(2n+3)}{a^{2n+3}} r_1^n b^{n+1} \\
&\times \sum_{m=0}^{n} \frac{\varepsilon_m (n-m)!}{(n+m)!} P_n^m(\cos\theta_1) \\
&\times \int_0^{\pi} \int_0^{2\pi} u(b,\theta,\varphi) P_n^m(\cos\theta) \cos m(\varphi - \varphi_1) \sin\theta \, d\theta \, d\varphi.
\end{aligned}$$

As we see from Theorem 2.1, the source functions ρ^* obtained are harmonic functions on $r < a$.

Indeed,

$$P_n^m(\cos\theta_1) r_1^n \cos m\varphi_1, \quad P_n^m(\cos\theta_1) r_1^n \sin m\varphi_1$$

are harmonic functions, and from (2.15) and (2.16)

$$|\varphi^*(\mathbf{r}_1) \leqq \|u\|_{H_{K_{i,a}}} \left\| \frac{1}{|\mathbf{r}-\mathbf{r}_1|} \right\|_{H_{K_{i,a}}}$$

$$= \|u\|_{H_{K_{i,a}}} \left\{ \sum_{n=0}^{\infty} \frac{(2n+1)(2n+3)r_1^{2n}}{a^{2n+3}} \right\}^{\frac{1}{2}} . \tag{2.17}$$

Hence, the expansion of $\rho^*(\mathbf{r}_1)$ in the right hand side in Theorem 2.1 converges absolutely and uniformly on $r \leqq a' < a$ for any fixed a'.

The fact that ρ^* are harmonic means that the source functions obtained are smooth and uniform on the ball $r < a$, in a sense.

In order to get a more general source, we shall consider the integral transform

$$u(\mathbf{r}') = \frac{1}{4\pi} \int_{r<a} \frac{1}{|\mathbf{r}'-\mathbf{r}|} \rho_1(\mathbf{r}) D(r) d\mathbf{r}$$

for a nonnegative measurable function $D(r)$ depending only on r such that the kernel form

$$K_{i,a}(\mathbf{r}', \mathbf{r}''; D(r)) = \frac{1}{4\pi} \int_{r<a} \frac{1}{|\mathbf{r}'-\mathbf{r}||\mathbf{r}''-\mathbf{r}|} D(r) d\mathbf{r}$$
$$(= K_{i,a}(D(r)))$$

exists for $r', r'' > a$. Then, we can obtain the inversion formula representing ρ_1^* in terms of $u(b, \theta, \varphi)$ satisfying

$$\|u\|^2_{H_{K_{i,a}(D(r))}} = \frac{1}{4\pi} \int_{r<a} \rho_1^*(\mathbf{r})^2 D(r) d\mathbf{r},$$

as in Theorem 2.1. Hence,

$$\rho(\mathbf{r}) = \rho_1^*(\mathbf{r}) D(r).$$

For this argument, see Section 3 for $D(r) = r^{-2}$. By taking a suitable $D(r)$, we will be able to obtain a more general and practical source $\rho(\mathbf{r})$. This technique is valid in the following situations, similarly.

3. CASE WITH UNBOUNDED SUPPORT ON $\mathbb{R}^3$

Next, we shall consider the case such that the supports of ρ are contained in the sphere $r > a$. In this case, in order to request the existence of the corresponding reproducing kernel we assume that

$$\int_{r>a} \rho(\mathbf{r})^2 r^2 d\mathbf{r} < \infty. \tag{3.1}$$

In order to examine the integral representation

$$u(\mathbf{r}') = \frac{1}{4\pi}\int_{r>a}\frac{1}{|\mathbf{r}'-\mathbf{r}|}\rho(\mathbf{r})d\mathbf{r} \tag{3.2}$$

of u, we shall consider the integral transform in the form

$$u(\mathbf{r}') = \frac{1}{4\pi}\int_{r>a}\frac{1}{|\mathbf{r}'-\mathbf{r}|}\rho_1(\mathbf{r})\frac{1}{r^2}d\mathbf{r} \tag{3.3}$$

for the functions $\rho_1 = r^2\rho$ satisfying

$$\int_{r>a}\rho_1(\mathbf{r})^2\frac{1}{r^2}d\mathbf{r} = \int_{r>a}\rho(\mathbf{r})^2 r^2 d\mathbf{r} < \infty. \tag{3.4}$$

We form the reproducing kernel

$$K_{o,a}(\mathbf{r}',\mathbf{r}'') = \frac{1}{4\pi}\int_{r>a}\frac{1}{|\mathbf{r}'-\mathbf{r}||\mathbf{r}''-\mathbf{r}|}\frac{1}{r^2}d\mathbf{r}. \tag{3.5}$$

By using (2.3) and the orthogonality of (2.5) and (2.6), we have

$$\begin{aligned} K_{o,a}(\mathbf{r}',\mathbf{r}'') &= \sum_{n=0}^{\infty}\frac{r'^n r''^n}{(2n+1)^2 a^{2n+1}}\sum_{m=0}^{n}\frac{\varepsilon_m(n-m)!}{(n+m)!} \\ &\times P_n^m(\cos\theta')P_n^m(\cos\theta'')(\cos m\varphi'\cos m\varphi'' + \sin m\varphi'\sin m\varphi''), \\ &\text{for}\quad r', r'' < a. \end{aligned} \tag{3.6}$$

Hence, by the parallel arguments to Theorem 2.1, we have

Theorem 3.1. For the source functions ρ^* satisfying (3.2) and (3.1) with the minimum norms in (3.1), we have the inverse formula, for any fixed $\hat{b}(0 < \hat{b} < a)$,

$$\begin{aligned} \rho^*(\mathbf{r}_1) &= \frac{1}{4\pi r_1^2}\sum_{n=0}^{\infty}(2n+1)^3 a^{2n+1} r_1^{-(n+1)}\hat{b}^{-n} \\ &\times \sum_{m=0}^{n}\frac{\varepsilon_m(n-m)!}{(n+m)!}P_n^m(\cos\theta_1)\int_0^{\pi}\int_0^{2\pi}u(\hat{b},\theta,\varphi) \\ &\times P_n^m(\cos\theta)\cos m(\varphi-\varphi_1)\sin\theta d\theta d\varphi. \end{aligned}$$

4. $\mathbb{R}^2$ CASE

We shall consider the 2-dimensional potential

$$u(\mathbf{r}') = \frac{1}{2\pi}\int_{r<a} \log\frac{1}{|\mathbf{r}'-\mathbf{r}|}\rho(\mathbf{r})d\mathbf{r} \tag{4.1}$$

for the source functions ρ satisfying

$$\int_{r<a} \rho(\mathbf{r})^2 d\mathbf{r} < \infty \tag{4.2}$$

whose supports are contained in the disc $r < a$.

In order to examine the integral transform (4.1) with (4.2), we form the reproducing kernel

$$K_{i,a}^{(2)}(\mathbf{r}', \mathbf{r}'') = \frac{1}{2\pi}\int_{r<a} \log\frac{1}{|\mathbf{r}'-\mathbf{r}|}\log\frac{1}{|\mathbf{r}''-\mathbf{r}|}d\mathbf{r}$$
$$\text{for}\quad r', r'' > a. \tag{4.3}$$

By using the expansion

$$\log\frac{1}{|\mathbf{r}'-\mathbf{r}|} = \log\frac{1}{r'} + \sum_{n=1}^{\infty}\frac{1}{n}(\frac{r}{r'})^n \cos n(\varphi'-\varphi),$$
$$\text{for}\quad r' > r \tag{4.4}$$

in the polar coordinates (r,φ) and (r',φ') (cf. [6, p. 1188]), we have

$$K_{i,a}^{(2)}(\mathbf{r}', \mathbf{r}'') = \frac{a^2}{2}\log\frac{1}{r'}\log\frac{1}{r''} + \frac{1}{4}$$
$$\times\sum_{n=1}^{\infty}\frac{a^{2n+2}}{n^2(n+1)}\frac{1}{r'^n r''^n}(\cos n\varphi' \cos n\varphi'' + \sin n\varphi' \sin n\varphi'')$$
$$\text{for}\quad r', r'' > a, \tag{4.5}$$

which converges absolutely. Hence, as in Theorem 3.1 we have

Theorem 4.1. The source functions ρ^* with the minimum norms (4.2) satisfying (4.1) for $r' > a$ are expressible in the form, for any fixed $b (b > a)$

$$\rho^*(\mathbf{r}_1) = \left(u(\mathbf{r}), \log\frac{1}{|\mathbf{r}-\mathbf{r}_1|}\right)_{H_{K_{i,a}^{(2)}}}$$
$$= \frac{1}{\pi a^2 \log\frac{1}{b}}\int_0^{2\pi} u(b,\varphi)d\varphi$$
$$+ \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{n(n+1)b^n r_1^n}{a^{2n+2}}\int_0^{2\pi} u(b,\varphi)\cos n(\varphi-\varphi_1)d\varphi.$$

Next, we shall examine the case such that ρ have unbounded supports. In order to consider the potential

$$u(\mathbf{r}') = \frac{1}{2\pi}\int_{r>a} \log\frac{1}{|\mathbf{r}'-\mathbf{r}|}\rho(\mathbf{r})d\mathbf{r}, \tag{4.6}$$

we shall examine the integral transform

$$u(\mathbf{r}') = \frac{1}{2\pi}\int_{r>a} \log\frac{1}{|\mathbf{r}'-\mathbf{r}|}\rho_1(\mathbf{r})\frac{1}{r^3}d\mathbf{r} \tag{4.7}$$

for the functions $\rho_1 = r^3\rho$ satisfying

$$\int_{r>a} \rho_1(\mathbf{r})^2\frac{1}{r^3}d\mathbf{r} = \int_{r>a}\rho(\mathbf{r})^2 r^3 d\mathbf{r} < \infty. \tag{4.8}$$

Then, the corresponding reproducing kernel $K^{(2)}_{o,a}(\mathbf{r}',\mathbf{r}'')$ can be calculated, using (4.4), as follows:

$$\begin{aligned} K^{(2)}_{o,a}(\mathbf{r}',\mathbf{r}'') &= \frac{1}{2\pi}\int_{r>a}\log\frac{1}{|\mathbf{r}'-\mathbf{r}|}\log\frac{1}{|\mathbf{r}''-\mathbf{r}|}\frac{1}{r^3}d\mathbf{r} \\ &= \frac{1}{a}\left\{2+(\log\frac{1}{a})^2 - 2\log\frac{1}{a}\right\} \\ &+ \frac{1}{2}\sum_{n=1}^{\infty}\frac{r'^n r''^n}{(2n+1)n^2a^{2n+1}}(\cos n\varphi'\cos n\varphi'' + \sin n\varphi'\sin n\varphi''). \end{aligned} \tag{4.9}$$

Hence, we have

Theorem 4.2. The source functions ρ^* satisfying (4.6) and (4.8) with the minimum norms are expressible in the form, for any fixed $\hat{b}(0 < \hat{b} < a)$

$$\begin{aligned} \rho^*(\mathbf{r}_1) &= \frac{1}{r_1^3}\left(u(\mathbf{r}), \log\frac{1}{|\mathbf{r}-\mathbf{r}_1|}\right)_{H_{K^{(2)}_{i,a}}} \\ &= \frac{1}{r_1^3}\Bigg[\left\{\frac{1}{a}(2+(\log\frac{1}{a})^2 - 2\log\frac{1}{a})\right\}^{-1} \\ &\times\left(\frac{1}{2\pi}\int_0^{2\pi}u(\hat{b},\varphi)d\varphi\right)\log\frac{1}{r_1} \\ &+ \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{n(2n+1)a^{2n+1}}{\hat{b}^n r_1^n}\int_0^{2\pi}u(\hat{b},\varphi)\cos n(\varphi-\varphi_1)d\varphi\Bigg]. \end{aligned}$$

5. $\mathbb{R}^1$ CASE

We shall consider the one dimensional potential in the form

$$u(x) = \int_{-a}^{a} -\frac{1}{2}|x-t|\rho(t)dt, x > a > 0 \tag{5.1}$$

for the source functions ρ satisfying

$$\int_{-a}^{a} \rho(t)^2 dt < \infty. \tag{5.2}$$

In order to examine the integral transform (5.1) with (5.2), we form the reproducing kernel

$$K_r(x,y) = \frac{1}{4}\int_{-a}^{a} |x-t||y-t|dt = \frac{a}{2}(xy + \frac{1}{3}a^2) \quad \text{for} \quad x, y > a. \tag{5.3}$$

This expression means that the Hilbert space H_{K_r} admitting the reproducing kernel $K_r(x,y)$ is two dimensional, and any members $f(x)$ of H_{K_r} are expressible in the form

$$f(x) = \frac{a}{2}(c_1 x + c_2 \frac{1}{3}a^2) \quad \text{for} \quad x > a \tag{5.4}$$

for some constants c_1 and c_2, and furthermore

$$\|f\|^2_{H_{K_r}} = \frac{a}{2}(c_1^2 + c_2^2 \frac{1}{3}a^2). \tag{5.5}$$

The constants c_1 and c_2 are determined by any two point values, say $f(b_1)$ and $f(b_2)$ for $b_1, b_2 > a$ as follows:

$$c_1 = \frac{2}{a}\frac{f(b_1) - f(b_2)}{b_1 - b_2} \tag{5.6}$$

and

$$c_2 = \frac{6}{a^3}\frac{b_1 f(b_2) - b_2 f(b_1)}{b_1 - b_2}. \tag{5.7}$$

We thus have

Theorem 5.1. In the potential (5.1) satisfying (5.2), the source functions ρ^* with the minimum norms in (5.2) are expressible in terms of $u(b_1)$ and $u(b_2)(b_1, b_2 > a, b_1 \neq b_2)$ in the form

$$\begin{aligned}\rho^*(t) &= (u(x), -\frac{1}{2}|x-t|)_{H_{K_r}} \\ &= \frac{3(b_1 u(b_2) - b_2 u(b_1))}{a^3(b_1 - b_2)}t - \frac{u(b_1) - u(b_2)}{a(b_1 - b_2)}.\end{aligned}$$

In order to determine source functions with unbounded supports we shall examine the integral transform, for $a > 0, x < a$

$$u(x) = \int_a^\infty -\frac{1}{2}|x-t|\rho(t)dt$$
$$= \int_a^\infty -\frac{1}{2}|x-t|\rho_1(t)\frac{1}{t^4}dt \tag{5.8}$$

for the source functions $\rho = \rho_1 t^{-4}$ satisfying

$$\int_a^\infty \rho_1(t)^2 \frac{dt}{t^4} = \int_a^\infty \rho(t)^2 t^4 dt < \infty. \tag{5.9}$$

We form the corresponding reproducing kernel

$$K_l(x,y) = \int_a^\infty -\frac{1}{2}|x-t| \cdot -\frac{1}{2}|y-t|\frac{dt}{t^4}$$
$$= \frac{1}{12a^3}(x-\frac{3}{2}a)(y-\frac{3}{2}a) + \frac{1}{16a}. \tag{5.10}$$

Hence, by the parallel arguments to Theorem 5.1, we have

Theorem 5.2. In the potential u in (5.8) satisfying (5.9), the source functions ρ^* with the minimum norms in (5.9) are expressible in terms of $u(b_1)$ and $u(b_2)$ for any fixed two points $b_1, b_2 < a$ as follows:

$$\rho^*(t) = \frac{1}{t^4}(u(x), -\frac{1}{2}|x-t|)_{H_{K_l}}$$
$$= \frac{1}{t^4}\Big[\frac{8a}{b_1-b_2}\Big\{(b_2u(b_1) - b_1u(b_2)) - \frac{3}{2}a(u(b_1) - u(b_2))\Big\}t$$
$$+ \frac{12a^2}{b_1-b_2}\{(b_1u(b_2) - b_2u(b_1)) + 2a(u(b_1) - u(b_2))\}\Big].$$

ACKNOWLEDGMENTS

The author wishes to express his sincere thanks to Professor Victor Isakov for his very valuable comments for the harmonicity of ρ^* in Theorem 2.1 at the 2nd International Symposium "Inverse Problems in Engineering Sciences "(Osaka, July 27-30, 1994) to the old version. For the harmonicity of ρ^*, Professor Fumi-Yuki Maeda kindly taught the author a more general proof using Weyl's lemma at the Meeting of "Integral Transforms and their Applications"at Kyoto University (August 25-26, 1994). The research was partially supported by the Japanese Ministry of Education, Science and Culture ; Grant-in-Aid General Research (C) 06640201.

REFERENCES

1. V.Isakov, "Inverse Source Problems," Mathematical Surveys and Monographs Number 34, American Mathematical Society, 1990.
2. P.M.Morse and H.Feshbach, "Methods of Theoretical Physics," McGraw-Hill Book Company Inc., New York Toronto London Tokyo, 1953.
3. S. Saitoh, *Hilbert spaces induced by Hilbert space valued functions*, Proc. Amer. Math. Soc. **89** (1983), 74-78.
4. S. Saitoh, "Theory of Reproducing Kernels and its Applications," Longman Scientific & Technical, Harlow, Essex, 1988.