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An error estimation of the reconstruction algorithm in computed tomography

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1 Introduction

1.1 Radon transformation

Let $f = f(x, y)$ be a piecewise continuous function on the plane with compact support,e.g., characteristic function supported on plane figures circumscribed by square, circle or asteroid. For any line L: $x\cos\theta + y\sin\theta = \xi$, let

$$\varphi(\theta,\xi) = \int_{-\infty}^{\infty} f(\xi\cos\theta + s\sin\theta, \xi\sin\theta - s\cos\theta)ds \tag{1.1}$$

where s is length measured along L. This function φ is the Radon transform of f. Let us write

$$\psi(\xi; x, y) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\theta, \xi + x\cos\theta + y\sin\theta)d\theta.$$

J. Radon [1],[2] gave the following inversion formula:

$$f(x, y) = -\frac{1}{\pi}\int_0^{\infty} \frac{\psi(\xi; x, y) - \psi(0; x, y)}{\xi^2}d\xi.$$

1.2 Approximation of the singular integral

Our problem is to make a good approximation to the singular integral

$$T(\psi) = \int_0^{\infty} \frac{\psi(\xi) - \psi(0)}{\xi^2}d\xi \tag{1.2}$$

where $\psi(\xi) = \psi(\xi; x, y)$.

Now, for step size $\delta > 0$, let us take

$$x_i = (i - 1/2)\delta, \quad I_i = [x_i, x_{i+1}), \quad i = 0, 1, 2, \ldots$$

and we set

$$\psi_\delta(\xi) = \psi(i\delta), \quad \xi \in I_i, \quad i = 0, 1, 2, \ldots.$$

We adopt $T(\psi_\delta)$ as an approximation of the singular integral (1.2). It is easily seen that

$$T(\psi_\delta) = \frac{1}{\delta}\{\sum_{i=1}^{\infty} \frac{\psi(i\delta)}{i^2 - 1/4} - 2\psi(0)\}$$

holds.

Our problem is to make a numerical integration formula for the singular integral $T(\psi)$. Moreover, we will investigate the order of accuracy of our integration formula when the function f is piecewise continuous.

2 Behavior of $\psi(\xi)$ near $\xi = 0$

2.1 Analytical forms of $\psi(\xi)$ near $\xi = 0$

Let us assume, for simplicity, $x = y = 0$ in (1.1).

For f, we set

$$m(r) = \frac{1}{2\pi}\int_0^{2\pi} f(r\cos\theta, r\sin\theta)d\theta, \quad r > 0.$$

Then it follows easily from (1.1) that

$$\psi(\xi) = 2\int_\xi^\infty \frac{r \cdot m(r)}{\sqrt{r^2 - \xi^2}}dr$$

holds.

Let us treat the case where $m(r)$ is in the following functional form :

$$m(r) = \begin{cases} ar^{\alpha-1} + b, & 0 < r < R \\ 0, & r > R. \end{cases} \tag{2.1}$$

where $\alpha > 0$, $a \neq 0$,b and $R > 0$ are constants. Let $\psi_\alpha(\xi)$ be the function $\psi(\xi)$ corresponding to the above $m(r)$ with α.

$$\begin{aligned} \psi_\alpha(\xi) &= \frac{a}{\pi}\int_\xi^R \frac{r^\alpha}{\sqrt{r^2 - \xi^2}}dr \quad (r = s\xi) \\ &= \frac{a\xi^\alpha}{\pi}\int_1^{R/\xi} \frac{s^\alpha}{\sqrt{s^2 - 1}}ds \\ &= \frac{a\xi^\alpha}{\pi}I_\alpha(\xi), \end{aligned} \tag{2.2}$$

where

$$I_\alpha = \int_1^{R/\xi} \frac{s^\alpha}{\sqrt{s^2-1}} ds. \tag{2.3}$$

Lemma 1 When $\alpha \neq 0$,

$$I_\alpha = \frac{1}{\alpha}\left(\frac{R}{\xi}\right)^\alpha \sqrt{1-\left(\frac{\xi}{R}\right)^2} + \frac{\alpha-1}{\alpha} I_{\alpha-2}(\xi).$$

Proof.

$$\frac{d}{ds}\left(\frac{1}{\alpha}\sqrt{s^{2\alpha}-s^{2\alpha-2}}\right) = \frac{s^\alpha}{\sqrt{s^2-1}} - \frac{\alpha-1}{\alpha}\frac{s^{\alpha-2}}{\sqrt{s^2-1}}.$$

∎

Lemma 2 When $\alpha < 0$,

$$I_\alpha(\xi) = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(-\alpha/2)}{\Gamma((1-\alpha)/2)} + \frac{1}{\alpha}\left(\frac{\xi}{R}\right)^{-\alpha} F(1/2, -\alpha/2; 1-\alpha/2; (\xi/R)^2). \tag{2.4}$$

Proof. By (2.3) with change of variables:

$$s = \frac{1}{\sqrt{u}}, \quad ds = -\frac{du}{2u^{3/2}}$$

we get

$$\begin{aligned} I_\alpha(\xi) &= \frac{1}{2}\int_{(\xi/R)^2}^1 u^{-1-\alpha/2}(1-u)^{-1/2} du \\ &= \frac{1}{2}\left(\int_0^1 u^{-1-\alpha/2}(1-u)^{-1/2} du - \int_0^{(\xi/R)^2} u^{-1-\alpha/2}(1-u)^{-1/2} du\right). \end{aligned} \tag{2.5}$$

For the first integral on the right-hand side of (2.5) we have

$$\int_0^1 u^{-1-\alpha/2}(1-u)^{-1/2} du = B(-\alpha/2, 1/2) = \frac{\sqrt{\pi}\Gamma(-\alpha/2)}{\Gamma(\frac{1-\alpha}{2})}. \tag{2.6}$$

For the second integral we have

$$\int_0^{(\xi/R)^2} u^{-1-\alpha/2}(1-u)^{-1/2} du = \left(\frac{\xi}{R}\right)^{-\alpha} \int_0^1 v^{-1-\alpha/2}(1-(\xi/R)^2 v)^{-1/2} dv,$$

and using Euler's integral representation of hypergeometric function [3], when $\alpha < 0$, we obtain

$$\int_0^1 v^{-1-\alpha/2}(1-(\xi/R)^2 v)^{-1/2} dv = -\frac{2}{\alpha} F(1/2, -\alpha/2; 1-\alpha/2; (\xi/R)^2).$$

Consequently we get (2.4) from (2.5) – (2.6) and the above expressions.

When $\alpha = 0$

$$I_0(\xi) = \int_1^{R/\xi} \frac{ds}{\sqrt{s^2-1}} = \log\left[\frac{R}{\xi}\left(1+\sqrt{1-(\xi/R)^2}\right)\right] \tag{2.7}$$

holds.

Now we can write ψ_α in explicit forms.

When $0 < \alpha < 2$ we have by (2.2) and Lemma 1

$$\psi_\alpha(\xi) = \frac{aR^\alpha}{\pi\alpha}\sqrt{1-(\xi/R)^2} + \frac{a(\alpha-1)}{\pi\alpha}\xi^\alpha I_{\alpha-2}(\xi),$$

and applying Lemma 2 to $I_{\alpha-2}$,

$$\begin{aligned}\psi_\alpha(\xi) &= \frac{aR^\alpha}{\pi\alpha}\sqrt{1-(\xi/R)^2} + \frac{a(\alpha-1)}{2\sqrt{\pi}\alpha}\cdot\frac{\Gamma((2-\alpha)/2)}{\Gamma((3-\alpha)/2)}\cdot\xi^\alpha \\ &+ \frac{a(\alpha-1)}{\pi\alpha(\alpha-2)}\cdot R^\alpha\cdot\left(\frac{\xi}{R}\right)^2\cdot F(1/2, 1-\alpha/2; 2-\alpha/2; (\xi/R)^2).\end{aligned}$$

Particularly for $\alpha = 1$ we have

$$\psi_1(\xi) = \frac{aR}{\pi}\sqrt{1-(\xi/R)^2}.$$

For $\alpha = 2$ we have by (2.2), Lemma 1 and (2.7).

$$\begin{aligned}\psi_2(\xi) &= \frac{aR^2}{2\pi}\sqrt{1-(\xi/R)^2} + \frac{a\xi^2}{2\pi}I_0(\xi) \\ &= \frac{aR^2}{2\pi}\sqrt{1-(\xi/R)^2} + \frac{a\xi^2}{2\pi}\log\left[\frac{\xi}{R}\left(1+\sqrt{1-(\xi/R)^2}\right)\right]\end{aligned}$$

Similarly when $2 < \alpha < 4$

$$\begin{aligned}\psi_\alpha(\xi) &= \frac{aR^\alpha}{\pi\alpha}\sqrt{1-(\xi/R)^2}\left(1+\frac{\alpha-1}{\alpha-2}\left(\frac{\xi}{R}\right)^2\right) \\ &+ \frac{a}{\pi}\frac{(\alpha-1)(\alpha-3)}{\alpha(\alpha-2)}\cdot\left[\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma((4-\alpha)/2)}{\Gamma((5-\alpha)/2)}\xi^\alpha\right. \\ &+ \left.\frac{R^\alpha}{\alpha-4}\left(\frac{\xi}{R}\right)^4\cdot F(1/2, 2-\alpha/2; 3-\alpha/2; (\xi/R)^2)\right]\end{aligned}$$

holds. When $\alpha = 4$

$$\psi_4(\xi) = \frac{aR^4}{4\pi}\sqrt{1-(\xi/R)^2}\left(1+\frac{3}{2}\left(\frac{\xi}{R}\right)^2\right) + \frac{aR^4}{\pi}\cdot\frac{3}{4\cdot 2}\cdot\log\left[\frac{\xi}{R}\left(1+\sqrt{1-(\xi/R)^2}\right)\right]$$

holds.

Thus we obtain the following functional forms of ψ_α for $\alpha > 0$:

$$\psi_\alpha(\xi) = \begin{cases} \text{const} \cdot \xi^\alpha + (\text{power series of } \xi^2) & \text{when } \alpha \neq \text{integer}, \\ (\text{power series of } \xi^2) & \text{when } \alpha \text{ is an odd integer}, \\ \text{const} \cdot \xi^\alpha \log \xi + (\text{power series of } \xi^2) & \text{when } \alpha \text{ is an even integer}. \end{cases}$$

2.2 Examples of α

Let us consider, as functions f to be reconstructed, the characteristic functions of the figures of square, disk and asteroid (see Figure 1.) When we take the dots in Figure 1 as reconstruction points, the corresponding α's are shown in Table 1.

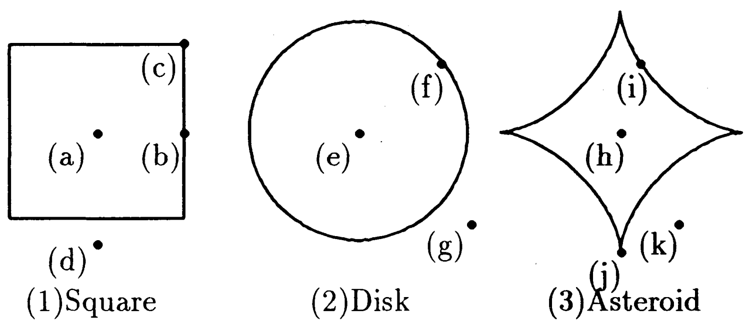

Figure 1: Examples of reconstruction points

Table 1: The values of α for reconstruction points.

figure	point	α
Square	(a) interior	1
	(b) edge	1
	(c) corner	1
	(d) exterior	1
Disk	(e) interior	1
	(f) edge	2
	(g) exterior	1
Asteroid	(h) interior	1
	(i) edge	2
	(j) cusp	3/2
	(k) exterior	1

For the above reconstruction points, we have

$$\frac{\psi(\xi)-\psi(0)}{\xi^2} = \begin{cases} (\text{power series of } \xi^2), & (\alpha=1) \\ C\log\xi + (\text{power series of } \xi^2), & (\alpha=2) \\ C\xi^{\alpha-2} + (\text{power series of } \xi^2), & (1<\alpha<2) \end{cases} \tag{2.8}$$

for $\xi > 0$ near the origin, where C is constant.

2.3 Examples of ψ

Let the reconstruction point be the origin $(0,0)$. In case of square, $\psi(\xi)$ is given by

$$\psi(\xi) = \begin{cases} 8\{\frac{l}{2}\log|\frac{\tan(\theta_0/2+\pi/4)}{\tan(\theta_0/2)}| + \xi\log|\tan\theta_0|\}, & |\xi| \le l/\sqrt{2} \\ 0, & |\xi| > l/\sqrt{2} \end{cases}$$

where $\theta_0 = \cos^{-1}(\sqrt{2}\xi/l) - \pi/4$, l is length of a side of the square.

In case of disk, $\psi(\xi)$ is given by

$$\psi(\xi) = \begin{cases} 4\pi\sqrt{r^2-\xi^2}, & |\xi| \le r \\ 0, & |\xi| > r. \end{cases}$$

where r is the radius of the disk.

3 On the accuracy of the numerical integration of the inverse Radon transformation

Let us set

$$\begin{aligned} a(\xi) &= \frac{\psi(\xi)-\psi(0)}{\xi^2}, & \chi_\delta(\xi) &= \begin{cases} \frac{1}{x_i x_{i+1}}, & \xi \in I_i (i=1,2,\ldots), \\ 0, & \xi \in I_0, \end{cases} \\ \bar{\psi}_\delta(\xi) &= \frac{1}{\delta}\int_{I_i}\psi(t)dt, \quad \xi \in I_i, & I_\delta &= \int_0^1 [\chi_\delta(\xi) - \frac{1}{\xi^2}]\cdot[\psi(\xi)-\psi(0)]d\xi, \quad \text{and} \\ \bar{\psi}_\delta(\xi) &= \psi(0), \quad \xi \in I_0, & J_\delta &= \int_1^\infty [\chi_\delta(\xi) - \frac{1}{\xi^2}]\cdot[\psi(\xi)-\psi(0)]d\xi. \\ T(\bar{\psi}_\delta) &= \int_\delta^\infty \frac{\bar{\psi}_\delta(\xi)-\psi(0)}{\xi^2}d\xi, \end{aligned}$$

Then, $T(\bar{\psi}_\delta) - T(\psi) = I_\delta + J_\delta$.

(A) $$|\chi_\delta(\xi)| \le \frac{3}{\xi^2}, \quad \xi > 0.$$

Proof. If $\xi = (i+\alpha)\delta, \alpha \le \frac{1}{2}$ then

$$\delta^2 \cdot \chi_\delta(\xi) = \frac{1}{i^2 - 1/4} \le \frac{3}{(i+\alpha)^2} = \frac{3\delta^2}{\xi^2}, \ i \ge 1.$$ ∎

(B) When $\xi \in I_i$ $(i \geq 1)$ we have $|\xi^2\chi_\delta(\xi) - 1| \leq \frac{1}{i-1/2}$.

Proof. If $\xi = (i+\alpha)\delta \in I_i$ then

$$\begin{aligned} \xi^2\chi_\delta(\xi) - 1 &= \frac{\xi^2}{x_i x_{i+1}} - 1 = \frac{(i+\alpha)^2}{i^2 - 1/4} - 1 \\ &= \frac{2\alpha i + \alpha^2 + 1/4}{i^2 - 1/4} \\ &= \frac{i + 1/2}{i^2 - 1/4} = \frac{1}{i - 1/2}. \end{aligned}$$

∎

(C) When $x_i \leq \xi < x_{i+1}$,

$$|\chi_\delta(\xi) - \frac{1}{\xi^2}| \leq \frac{\delta}{x_i^3} \quad (i \geq 1).$$

Proof. Let $x_i \leq \xi < x_{i+1}$, in view of (B),

$$|\chi_\delta(\xi) - \frac{1}{\xi^2}| \leq \frac{1}{(i-1/2)\xi^2} \leq \frac{1}{(i-1/2)x_i^2} = \frac{\delta}{x_i^3}$$

∎

(D) Let $1 \leq p' < \infty$. Then there exists a constant $C_{p'}$ independant of δ,

$$\int_0^1 |\xi^2\chi_\delta(\xi) - 1|^{p'}|d\xi \leq \begin{cases} C_1 \cdot \delta \log \frac{1}{\delta}, & p' = 1 \\ C_{p'} \cdot \delta, & 1 < p' < \infty \end{cases}$$

Proof. Let N be the maximum of integers which are less than $\frac{1}{\delta} + \frac{1}{2}$.

$$\begin{aligned} \int_0^1 |\xi^2\chi_\delta(\xi) - 1|^{p'}|d\xi &\leq \frac{\delta}{2} + \sum_{i=1}^{1/\delta - 1} \int_{x_i}^{x_{i+1}} |\xi^2\chi_\delta(\xi) - 1|^{p'}|d\xi \qquad \text{(from (B))} \\ &\leq \frac{\delta}{2} + \delta \sum_{i=1}^{N} \frac{1}{(i-1/2)^{p'}}, \end{aligned}$$

from which we can easily obtain the desired result. ∎

(E) Assume $a(\xi) \in L^p(0,1), 1 < p \leq \infty$. Then

$$|I_\delta| \leq \begin{cases} C_1 \cdot \delta \log \frac{1}{\delta} \cdot ||a||_{L^\infty(0,1)}, & p = \infty \\ C_{p'}^{1/p'} \cdot \delta^{1/p'} \cdot ||a||_{L^p(0,1)}, & 1 < p < \infty \end{cases}$$

Proof.

$$\begin{aligned} |I_\delta| &= |\int_0^1 (\xi^2\chi_\delta(\xi) - 1)a(\xi)d\xi| \\ &\leq (\int_0^1 |\xi^2\chi_\delta(\xi) - 1|^{p'})^{1/p'} \cdot ||a||_{L^p(0,1)} \end{aligned}$$

Then (E) follows from (D) and the above inequality. ∎

(F) When $\psi \in L^1(1,\infty)$ and $(i_0 - 1/2)\delta = 1$ for some integer i_0, then we have

$$|J_\delta| \leq \delta \cdot ||\psi||_{L^1(1,\infty)}.$$

Proof.

$$\begin{aligned} |J_\delta| &= |\int_1^\infty [\chi_\delta(\xi) - \frac{1}{\xi^2}] \cdot [\psi(\xi) - \psi(0)] d\xi| \\ &= |\int_1^\infty [\chi_\delta(\xi) - \frac{1}{\xi^2}] \psi(\xi) d\xi| \\ &\leq \delta \cdot \int_1^\infty |\psi(\xi)| d\xi \qquad \text{(by (C))}. \end{aligned}$$

∎

We summarize:

Theorem 1 When $a(\xi) \in L^p(0,1), \psi(\xi) \in L^1(1,\infty)$, and $\frac{1}{p} + \frac{1}{p'} = 1$ $(1 < p \leq \infty)$, we have

$$|T(\bar{\psi}_\delta) - T(\psi)| \leq \begin{cases} C\delta(\log \frac{1}{\delta} \cdot ||a||_{L^\infty(0,1)} + ||\psi||_{L^1(1,\infty)}), & p = \infty \\ C\delta^{1/p'}(||a||_{L^p(0,1)} + \delta^{1/p}||\psi||_{L^1(1,\infty)}), & 1 < p < \infty \end{cases} \tag{3.1}$$

where C is a constant independent of ψ and δ.

When $p = 1$ we have

Theorem 2 When $a(\xi) \in L^1(0,1), \psi(\xi) \in L^1(1,\infty)$,

$$T(\bar{\psi}_\delta) \to T(\psi) \quad (\delta \to 0).$$

Next we will estimate $T(\bar{\psi}_\delta - \psi_\delta)$.

For $\delta > 0$, we set $\xi_0 = i_0\delta, \quad 0 < \xi_0 < \xi_\infty, \quad$ (i_0 is integer).

We suppose that $\psi \in C^2[0,\xi_\infty)$ satisfies $\psi'(0) = 0$ and

$$\psi''(\xi_\infty - \eta) = a(\eta)\eta^\alpha, \quad 0 < \eta < \xi_\infty$$

where α is a negative constant, $a \in C^0[0,\xi_\infty]$. We set $\psi(\xi) = \psi(-\xi), \quad \xi < 0$, if necessary.

Then we have

Lemma 3.

$$T(\bar{\psi}_\delta - \psi_\delta) = O(\delta)$$

holds, when $\alpha \geq -2$.

Proof. For δ let

$$\begin{aligned} I_i &= ((i-1/2)\delta, (i+1/2)\delta), \\ i_\infty &= \max\{i : (i+1/2)\delta \leq \xi_\infty\}. \end{aligned}$$

Using Taylor's theorem, we can get

$$\psi(i\delta + \eta) = \psi(i\delta) + \eta\psi'(i\delta) + \frac{\eta^2}{2}\psi''(i\delta + \theta)$$

where $0 < \theta < 1$. Thereby

$$\begin{aligned} \bar{\psi}_\delta(i\delta) &= \frac{1}{\delta}\int_{I_i} \psi(\xi)d\xi \\ &= \psi(i\delta) + \frac{\delta^2}{24}\psi''(i\delta + \theta), \quad |\theta| < \frac{1}{2}. \end{aligned}$$

Let T_0,T_1 be defined by the relations

$$T(\bar{\psi}_\delta - \psi_\delta) = T_0 + T_1,$$

$$\begin{aligned} T_0 &= \frac{1}{\delta}\sum_{i=1}^{i_0} \frac{\bar{\psi}(i\delta) - \psi(i\delta)}{i^2 - 1/4} - \frac{2}{\delta}[\bar{\psi}(0) - \psi(0)], \\ T_1 &= \frac{1}{\delta}\sum_{i=i_0+1}^{i_\infty - 1} \frac{\bar{\psi}(i\delta) - \psi(i\delta)}{i^2 - 1/4}. \end{aligned}$$

(i) Estimate of T_1

First we have

$$T_1 = \frac{\delta}{24}\sum_{i=i_0+1}^{i_\infty - 1} \frac{\psi''(i\delta + \theta\delta)}{i^2 - 1/4}$$

and, when $i \leq i_\infty - 1$,

$$\begin{aligned} |\psi''(i\delta) + \theta\delta)| &\leq ||a||_\infty |\xi_\infty - i\delta - \theta\delta|^\alpha \\ &\leq ||a||_\infty |\xi_\infty - \frac{\delta}{2} - i\delta|^\alpha \end{aligned}$$

therefore

$$\begin{aligned} |T_1| &\leq \frac{\delta}{24}||a||_\infty \sum_{i=i_0+1}^{i_\infty - 1} \frac{|\xi_\infty - \delta/2 - i\delta|^\alpha}{i^2 - 1/4} \\ &\leq \frac{\delta}{12}||a||_\infty \sum_{i=i_0+1}^{i_\infty - 1} \frac{|\xi_\infty - \delta/2 - i\delta|^\alpha}{i^2} \\ &= \frac{\delta^{3+\alpha}}{12\xi_0^2}||a||_\infty \sum_{i=i_0+1}^{i_\infty - 1} |\xi_\infty - \delta/2 - i\delta|^\alpha. \end{aligned}$$

If we set

$$A = \frac{\xi_\infty}{\delta} + \frac{1}{2} - i_\infty, B = \frac{\xi_\infty}{\delta} - \frac{3}{2} - i_0,$$

then we have

$$\sum_{i=i_0+1}^{i_\infty - 1} |\frac{\xi_\infty}{\delta} - \frac{1}{2} - i|^\alpha \le \int_A^B x^\alpha dx + A^\alpha.$$

Hence we obtain

$$|T_1| \le \frac{\delta^{3+\alpha}}{12\xi_0} ||a||_\infty \left[\frac{1}{1+\alpha}(B^{1+\alpha} - A^{1+\alpha}) + A^\alpha\right]$$

holds when $\alpha \neq -1$. Since

$$\delta A = \xi_\infty + \frac{\delta}{2} - i_\infty \delta \sim \text{const} \cdot \delta, \quad (3.2)$$

$$\delta B = \xi_\infty - \frac{3}{2}\delta - \xi_0 \sim \xi_\infty - \xi_0, \quad (3.3)$$

we have

$$\delta^{1+\alpha}\left[\frac{1}{1+\alpha}(B^{1+\alpha} - A^{1+\alpha}) + A^\alpha\right] \sim \begin{cases} \text{const} & (1+\alpha > 0) \\ \text{const} \cdot \delta^{1+\alpha} & (1+\alpha < 0) \end{cases}$$

Accordingly

$$T_1 = \begin{cases} O(\delta^2) & (1+\alpha > 0) \\ O(\delta^{3+\alpha}) & (1+\alpha < 0) \end{cases}.$$

When $\alpha = 1$ we have

$$|T_1| \le \frac{\delta^2}{12\xi_0^2} ||a||_\infty \left[\log\frac{B}{A} + \frac{1}{A}\right].$$

Also using (3.2) and (3.3) we can show that

$$T_1 = O(\delta^2 \log\frac{1}{\delta}).$$

(ii) Estimate of T_0

$$\frac{1}{\delta}\sum_{i=1}^{i_0} \frac{|\bar{\psi}(i\delta) - \psi(i\delta)|}{i^2 - 1/4} \le \frac{1}{12\delta}\sum_{i=1}^{i_0}\frac{|\bar{\psi}(i\delta) - \psi(i\delta)|}{i^2}$$
$$\le \frac{\delta}{12} \cdot ||\psi''||_{L^\infty}\frac{\pi^2}{6}$$

and

$$\frac{2}{\delta}[\bar{\psi}(0) - \psi(0)] = \frac{\delta}{12}\psi''(\theta\delta) = O(\delta)$$

therefore

$$T_0 = O(\delta).$$

Hence we have

$$T(\bar{\psi}_\delta - \psi_\delta) = O(\delta).$$

We note that the statements of Theorems 1 and 2 are valid if $\bar{\psi}_\delta$ is replaced by ψ_δ, when ψ satisfies conditions of Lemma 3 and when $\alpha \geq -2$.

4 Results of numerical experiments

On our algorithm, the reconstruction was performed for characteristic functions f of figures of square, disk or asteroid.

For each reconstruction point, the absolute error of reconstruction is proportional to δ^k when δ is in a certain interval. We show graphs of absolute error in our numerical experiments. Here the abscissa is the logarithm of δ and the ordinate is the logarithm of absolute error. (See Figure 2 – Figure 4.)

In the following table (Table 2) k and k' denote the orders of the accuracy of reconstruction obtained in the numerical experiments and theoretically in the right-hand side of the inequality (3.1),respectively. To get k' we have calculated the number α in (2.1) and used (2.8) in each case. We note that the order k' is obtained without regard to the differentiability of ψ. Therefore we can only expect that $k \geq k'$ holds.

Table 2: Results of numerical experiments.

f	reconstruction point		k	k'	α
square	center	(a)	3	1	1
	edge point	(b)	3	1	1
	corner	(c)	3	1	1
	exterior point	(d)	3	1	1
circle	center	(e)	1.5	1	1
	edge point	(f)	1	$k' < 1$	2
	exterior point	(g)	2	1	1
asteroid	center	(h)	2	1	1
	edge point	(i)	1	$k' < 1$	2
	cusp	(j)	0.5	$k' < 0.5$	3/2
	exterior point	(k)	2	1	1

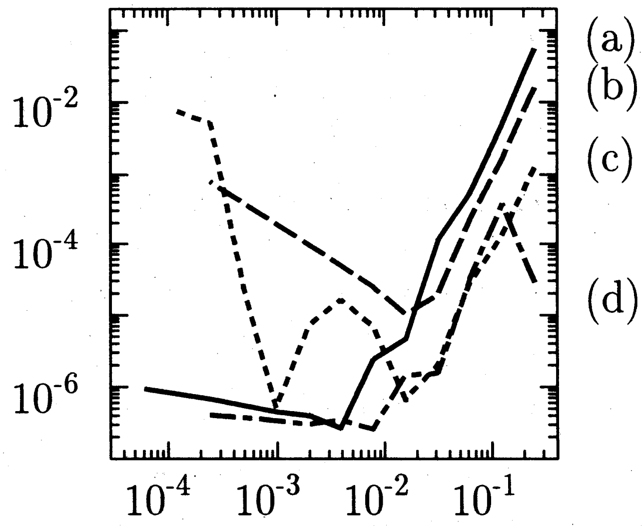

Figure 2:Square

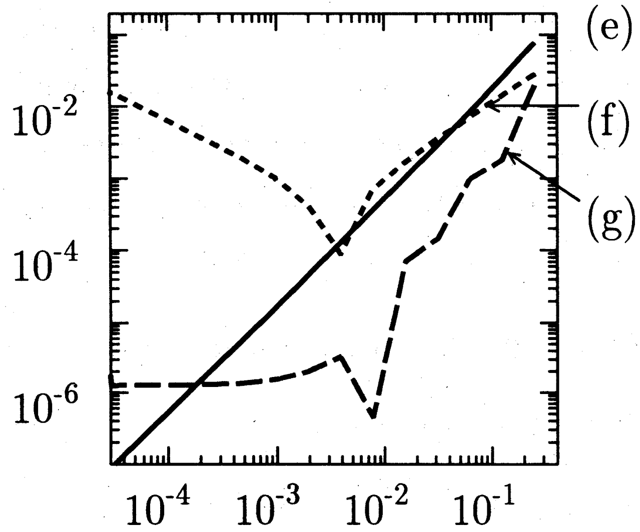

Figure 3:Disk

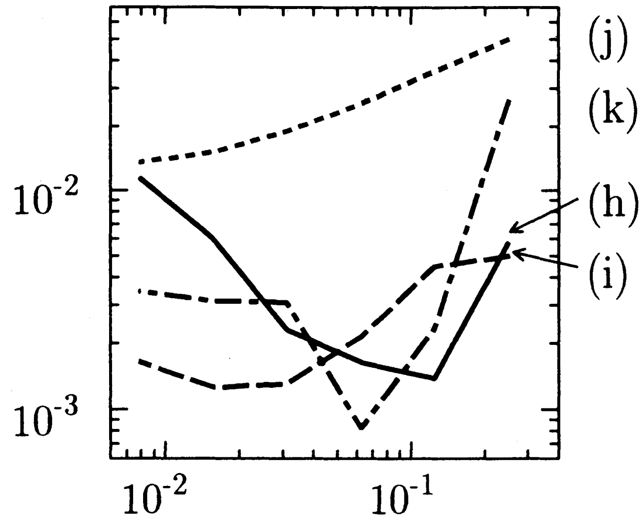

Figure 4:Asteroid

Finally let us write a summary of our results.

(i) We have tried to make a numerical reconstruction formula for piecewise continuous functions. We have adopted a kind of rectangular rule for the inverse Radon transformation.

(ii) In numerical experiments we have obtained orders of the accuracy of reconstruction which are greater than or equal to orders assured by the L^p error estimates.

(iii) We have order $k = 0.5$ for the cusp of the asteroid which is the most difficult to reconstruct in our experiments, and we have order $2 \sim 3$ for ordinary continuous points.

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