

## STRONG CONVERGENCE THEOREMS FOR NONEXPANSIVE MAPPINGS IN BANACH SPACE

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ABSTRACT. We prove for a nonexpansive mapping  $T$  that under certain conditions the strong  $\lim_{t \rightarrow 1^-} G_t(x)$  exists and is a fixed point of  $T$ , where  $G_t(x) = (1-t)x + tTG_t(x)$ ,  $0 \leq t < 1$ .

### 1. Introduction

Let  $C$  be a nonempty closed convex subset of a Banach space  $E$ . A mapping  $T : C \rightarrow C$  is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all  $x, y$  in  $C$ .

Let  $E^*$  be the dual space of  $E$ . Then the value of  $f \in E^*$  at  $x \in E$  will be denoted by  $\langle x, f \rangle$ . With each  $x \in E$ , we associate the set

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}.$$

Using the Hahn-Banach theorem, it is immediately clear that  $J(x) \neq \emptyset$  for each  $x \in E$ . The multivalued operator  $J : E \rightarrow E^*$  is called the duality mapping of  $E$ . Let  $B = \{x \in E : \|x\| = 1\}$  be the unit sphere of  $E$ . Then the norm of  $E$  is said to be Gateaux differentiable (and  $E$  is said to be smooth) if

$$\lim_{r \rightarrow 0} \frac{\|x + ry\| - \|x\|}{r}$$

exists for each  $x$  and  $y$  in  $B$ . It is said to be Frechet differentiable if for each  $x$  in  $B$ , this limit is attained uniformly for  $y$  in  $B$ . Finally, it is said to be uniformly Frechet differentiable (and  $E$  is said to be uniformly smooth) if the limit is attained uniformly for  $(x, y)$  in  $B \times B$ . It is well known that if  $E$  is smooth, then the duality mapping  $J$  is single valued. It is also known that if  $E$  has a Frechet differentiable norm, then  $J$  is norm to norm continuous.

The purpose of this note is to continue the discussion concerning the strong convergence of the path  $t \rightarrow G_t(x)$ ,  $0 \leq t < 1$  defined by (1) below for each  $x$  in  $C$ . We prove for a nonexpansive mapping  $T$  that under certain conditions the strong  $\lim_{t \rightarrow 1^-} G_t(x)$  exists and is a fixed point of  $T$ . The first results of this nature were established by Brower([2]) and Browder and Petryshyn([4]).

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## 2. Lemmas

Let  $E$  be a Banach space. Then the modulus of convexity of  $E$  is defined as  $\delta_E(\varepsilon) = \inf\{1 - \frac{1}{2}\|x + y\| : x, y \in B_E \text{ and } \|x - y\| \geq \varepsilon\}$ , where  $B_E = \{x \in E : \|x\| \leq 1\}$  is the closed unit ball of  $E$ . We recall that  $E$  is said to have the modulus of convexity of power type  $p \geq 2$  (and  $E$  is said to be  $p$ -uniformly convex) if there exists a constant  $c > 0$  such that

$$\delta_E(\varepsilon) \geq c\varepsilon^p$$

for  $0 < \varepsilon \leq 2$ .

We now define the mapping  $G_t : C \rightarrow C$  by

$$G_t(x) = (1 - t)x + tTG_t(x) \quad (1)$$

for all  $x$  in  $C$  and  $0 \leq t < 1$ . It is clear that for each  $0 \leq t < 1$ , the fixed point set of  $G_t$  coincides with that of  $T$ .

We also recall that a Banach limit LIM is a bounded linear functional on  $\ell^\infty$  of norm 1 such that

$$\liminf_{n \rightarrow \infty} x_n \leq \text{LIM}\{x_n\} \leq \limsup_{n \rightarrow \infty} x_n$$

and

$$\text{LIM}\{x_n\} = \text{LIM}\{x_{n+1}\}$$

for all  $\{x_n\}$  in  $\ell^\infty$ .

**LEMMA 1.** (Prus and Smarzewski [6]) *Let  $E$  be a  $p$ -uniformly convex Banach space ( $p > 1$ ). Then there exists a constant  $c > 0$  such that*

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - cW_p(\lambda)\|x - y\|^p \quad (2)$$

for all  $x, y \in E$  and  $\lambda \in [0, 1]$ , where  $W_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$ .

**LEMMA 2.** *Let  $C$  be a nonempty closed convex and bounded subset of a  $p$ -uniformly convex Banach space  $E$ , and let  $\{x_n\}$  be a bounded sequence in  $E$ . We define the functional  $r : C \rightarrow \mathbb{R}$  by the formula*

$$r(x) = \text{LIM}\{\|x_n - x\|^p\}.$$

Then  $r(\cdot)$  is continuous and convex.

*Proof.* For  $x, y \in C$ , we have

$$\left| \|x_n - x\|^p - \|x_n - y\|^p \right| \leq p(\text{diam}C)^{p-1} \left| \|x_n - x\| - \|x_n - y\| \right|$$

and

$$\begin{aligned} |r(x) - r(y)| &= |\text{LIM}\{\|x_n - x\|^p\} - \text{LIM}\{\|x_n - y\|^p\}| \\ &\leq p(\text{diam}C)^{p-1} \text{LIM}\{|\|x_n - x\| - \|x_n - y\||\} \\ &\leq p(\text{diam}C)^{p-1} \text{LIM}\{\|x - y\|\} \\ &\leq p(\text{diam}C)^{p-1} \|x - y\|. \end{aligned}$$

For any fixed  $n \in N$  and  $0 < t < 1$ , by the inequality (2), we get

$$\begin{aligned} \|x_n - ((1-t)x + ty)\|^p &= \|(1-t)(x_n - x) + t(x_n - y)\|^p \\ &\leq (1-t)\|x_n - x\|^p + t\|x_n - y\|^p - cW_p(t)\|x - y\|^p \\ &\leq (1-t)\|x_n - x\|^p + t\|x_n - y\|^p. \end{aligned}$$

Taking the Banach limit LIM on each side, we obtain

$$\text{LIM}\{\|x_n - ((1-t)x + ty)\|^p\} \leq (1-t)\text{LIM}\{\|x_n - x\|^p\} + t\text{LIM}\{\|x_n - y\|^p\}.$$

Therefore we get

$$r((1-t)x + ty) \leq (1-t)r(x) + tr(y).$$

LEMMA 3. Let  $C$  be a nonempty closed convex subset of a  $p$ -uniformly convex and uniformly smooth Banach space  $E$ . Let  $\{x_n\}$  be a bounded sequence in  $E$ . Then for  $z_0 \in C$ ,

$$\text{LIM}\{\|x_n - z_0\|^p\} = \min_{y \in C} \text{LIM}\{\|x_n - y\|^p\}$$

if and only if

$$\text{LIM}\{\langle z - z_0, J(x_n - z_0) \rangle\} \leq 0$$

for all  $z \in C$ .

*Proof.* We first assume that  $\text{LIM}\{\|x_n - z_0\|^p\} = \min_{y \in C} \text{LIM}\{\|x_n - y\|^p\}$ . For  $z \in C$  and  $\lambda: 0 \leq \lambda \leq 1$ , we have

$$\begin{aligned} \|x_n - z_0\|^p &= \|x_n - \lambda z_0 - (1-\lambda)z + (1-\lambda)(z - z_0)\|^p \\ &\geq \|x_n - \lambda z_0 - (1-\lambda)z\|^p \\ &\quad + p(1-\lambda) \langle z - z_0, J(x_n - \lambda z_0 - (1-\lambda)z) \rangle \end{aligned}$$

since  $J(x)$  is subdifferential of the convex function  $\frac{1}{p}\|x\|^p$  ([3,p97]). Let  $\varepsilon > 0$  be given. Since  $E$  is uniformly smooth, the duality map is uniformly continuous on bounded subsets of  $E$  from the strong topology of  $E$  to the weak\* topology of  $E^*$  ([3]). Therefore,

$$|\langle z - z_0, J(x_n - \lambda z_0 - (1-\lambda)z) - J(x_n - z_0) \rangle| < \varepsilon$$

if  $\lambda$  is close enough to 1. Consequently, we have

$$\begin{aligned} \langle z - z_0, J(x_n - z_0) \rangle &> \varepsilon + \langle z - z_0, J(x_n - \lambda z_0 - (1-\lambda)z) \rangle \\ &\leq \varepsilon + \frac{1}{p(1-\lambda)} \{ \|x_n - z_0\|^p \\ &\quad - \|x_n - \lambda z_0 - (1-\lambda)z\|^p \}. \end{aligned}$$

Therefore, we have

$$\text{LIM}\{\langle z - z_0, J(x_n - z_0) \rangle\} \leq 0$$

for all  $z \in C$ .

To prove reverse, let  $z \in C$ . Then since

$$\|x_n - z\|^p - \|x_n - z_0\|^p \geq p \langle z_0 - z, J(x_n - z_0) \rangle$$

for all  $n \in N$  and  $\text{LIM}\{\langle z - z_0, J(x_n - z_0) \rangle\} \leq 0$  for all  $z \in C$ , we have

$$\text{LIM}\{\|x_n - z_0\|^p\} = \min_{z \in C} \text{LIM}\{\|x_n - z\|^p\}.$$

LEMMA 4. Let  $C$  be a closed convex and bounded subset of a  $p$ -uniformly convex and uniformly smooth Banach space  $E$ , and  $\{x_n\}$  be a bounded sequence of  $E$ . Then, the set

$$M = \{u \in C : LIM\{\|x_n - u\|^p\} = \min_{z \in C} LIM\{\|x_n - z\|^p\}\}$$

consists of one point.

*Proof.* Let  $g(z) = LIM\{\|x_n - z\|^p\}$  for every  $z \in C$  and  $r = \inf\{g(z) : z \in C\}$ . Then, by Lemma 2, the function  $g$  on  $C$  is convex and continuous and  $g(z) \rightarrow \infty$  as  $\|z\| \rightarrow \infty$ . From [1,p79], there exists  $u \in C$  with  $g(u) = r$ . Therefore  $M$  is nonempty. By Lemma 3, we know that  $u \in M$  if and only if  $LIM\{\langle z - u, J(x_n - u) \rangle\} \leq 0$  for all  $z \in C$ . We show that  $M$  consists of one point. Let  $u, v \in M$  and suppose  $u \neq v$ . Then by [7, Theorem 1], there exists a positive number  $\varepsilon$  such that

$$\langle x_n - u - (x_n - v), J(x_n - u) - J(x_n - v) \rangle \geq \varepsilon$$

for every  $n \in N$ . Therefore

$$LIM\{\langle v - u, J(x_n - u) - J(x_n - v) \rangle\} \geq \varepsilon > 0.$$

On the other hand, since  $u, v \in M$ , we have

$$LIM\{\langle u - v, J(x_n - v) \rangle\} < 0$$

and

$$LIM\{\langle v - u, J(x_n - u) \rangle\} < 0.$$

Then we have

$$LIM\{\langle v - u, J(x_n - u) - J(x_n - v) \rangle\} < 0.$$

This is a contradiction. Therefore  $u = v$ .

### 3. Main Results

THEOREM 1. Let  $C$  be a closed convex and bounded subset of a  $p$ -uniformly convex and uniformly smooth Banach space  $E$ , and  $\{x_n\}$  be a bounded sequence of  $E$  such that  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ . If  $T : C \rightarrow C$  is a nonexpansive mapping, then

$$M = \{u \in C : LIM\{\|x_n - u\|^p\} = \min_{z \in C} LIM\{\|x_n - z\|^p\}\}$$

is a fixed point set of  $T$ .

*Proof.* We will show that the set  $M$  is invariant under  $T$ . In fact, since  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ , we have, for  $u \in M$ ,

$$\begin{aligned} LIM\{\|x_n - Tu\|^p\} &= LIM\{\|Tx_n - Tu\|^p\} \\ &\leq LIM\{\|x_n - u\|^p\} \end{aligned}$$

and hence  $Tu \in M$ . On the other hand, by Lemma 4, we know that  $M$  consists of one point. Therefore this point is a fixed point of  $T$  and  $M$  is a fixed point set of  $T$ .

It is well known in ([8]) that a uniform smooth space has normal structure. Since such a space is also reflexive, each bounded closed convex subset of it has the fixed point property for nonexpansive mappings ([5]).

**THEOREM 2.** Let  $C$  be a closed convex and bounded subset of a  $p$ -uniformly convex and uniformly smooth Banach space  $E$ ,  $T : C \rightarrow C$  a nonexpansive mapping, and  $G_t : C \rightarrow C$ ,  $0 < t < 1$ , the family of mappings defined by (1). Then, for each  $x$  in  $C$ , the strong  $\lim_{t \rightarrow 1^-} G_t(x)$  exists and is a fixed point of  $T$ .

*Proof.* Note that from the preceding statement  $T$  has a fixed point in  $C$ . Let  $w$  be a fixed point of  $T$ . Fix a point  $x$  in  $C$ , denote  $G_t(x)$  by  $y(t)$ . Since  $y(t) - w = (1 - t)(x - w) + t(Ty(t) - Tw)$ ,

$$\|y(t) - w\| \leq \|x - w\|$$

and  $\{y(t)\}$  remains bounded as  $t \rightarrow 1^-$ . We also have

$$\begin{aligned} \lim_{t \rightarrow 1^-} \|y(t) - Ty(t)\| &= \lim_{t \rightarrow 1^-} \|(1 - t)x - (1 - t)Ty(t)\| \\ &= 0. \end{aligned}$$

Now let  $t_n \rightarrow 1^-$  and  $y_n = y(t_n)$ . Define  $f : C \rightarrow [0, \infty)$  by  $f(z) = \text{LIM}\{\|y_n - z\|^p\}$ . From Lemma 2  $f$  is continuous and convex,  $f(z) \rightarrow \infty$  as  $\|z\| \rightarrow \infty$ , which implies that  $f$  attains its infimum over  $C$ . That is, there exists a  $z_0 \in C$  such that

$$\text{LIM}\{\|y_n - z_0\|^p\} = \min_{y \in C} \text{LIM}\{\|y_n - y\|^p\}.$$

Let  $M$  be the set of minimizers of  $T$ . By Theorem 1,  $z_0 \in M$  is the fixed point of  $T$ . Therefore

$$\begin{aligned} \langle y_n - Ty_n, J(y_n - z_0) \rangle &= \langle y_n - Tz_0 + Tz_0 - Ty_n, J(y_n - z_0) \rangle \\ &= \|y_n - Tz_0\|^2 - \langle Ty_n - Tz_0, J(y_n - z_0) \rangle \\ &\geq \|y_n - Tz_0\|^2 - \|Ty_n - Tz_0\| \|y_n - z_0\| \\ &\geq \|y_n - Tz_0\|^2 - \|y_n - Tz_0\|^2 = 0 \end{aligned}$$

for all  $n$ . It follows that for  $x \in C$ ,

$$\begin{aligned} 0 &\leq \langle y_n - Ty_n, J(y_n - z_0) \rangle \\ &= \langle (1 - t_n)x + t_nTy_n - Ty_n, J(y_n - z_0) \rangle \\ &= \langle (1 - t_n)x - (1 - t_n)Ty_n, J(y_n - z_0) \rangle \\ &= (1 - t_n) \langle x - Ty_n, J(y_n - z_0) \rangle \end{aligned}$$

for all  $n$ . Thus, we get for  $x \in C$ ,

$$\langle y_n - x, J(y_n - z_0) \rangle \leq 0 \tag{3}$$

for all  $n$ . From Lemma 3

$$\text{LIM}\{\langle z - z_0, J(y_n - z_0) \rangle\} \leq 0 \tag{4}$$

for all  $z \in C$ . Choosing  $z = y_n$  in (4), we conclude that

$$\text{LIM}\{\|y_n - z_0\|\} \leq 0.$$

Thus there is a subsequence of  $\{y_n\}$  which converges strongly to  $z_0$ . To complete the proof, suppose that  $y_{n_k} \rightarrow z_1$  and  $y_{m_k} \rightarrow z_2$ . Then by (3),

$$\langle z_1 - x, J(z_1 - z_2) \rangle \leq 0$$

and

$$\langle z_1 - z_2, J(z_1 - z_2) \rangle \leq 0.$$

Hence  $z_1 = z_2$  and the strong  $\lim_{t \rightarrow 1^-} y(t)$  exists, which completes the proof.

### References

1. V. Barbu and Th. Precupanu, *Convexity and Optimization in Banach spaces*, Editura Academiei R.S.R., Bucharest (1978).
2. F. E. Browder, *Convergence of approximants to fixed points of nonexpansive mappings in Banach spaces*, Archs Ration. Mech. Anal. **24** (1967), 82-90.
3. F. E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces*, American Mathematical Society **18(Part 2)** (1976).
4. F. E. Browder and W. V. Petryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull. Amer. Math. Soc. **72** (1988), 571-575.
5. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), 1004-1006.
6. B. Prus and R. Smarzewski, *Strongly unique best approximations and centers in uniformly convex spaces*, J. Math. Anal. Appl. **121** (1987), 10-21.
7. J. Prus, *A characterization of uniform convexity and applications to accretive operators*, Hiroshima J. Math. **11** (1981), 229-234.
8. B. Turett, *A dual view of a theorem of Baillon*, Marcel Dekker, New York **80** (1982), 279-286.

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