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On a Relationship between Ekeland's Algorithm and Infimal Convolutions*

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Abstract. The aim of this paper is to give a study on a relationship between Ekeland's variational principle and infimal convolution. By help of graphic interpretation of infimal convolution, Ekeland's variational principle can be regarded as an algorithm similar to the proximal point algorithm for a problem of minimizing a lower semicontinuous proper convex function on a Hilbert space.

Key Words. Ekeland's variational principle, infimal convolution, convex analysis, optimization.

1. INTRODUCTION

Recently, good books related to optimization theory have been published; [2, 10, 14], in which Ekeland's variational principle and fixed point theorems are observed, but not devoted to relationship between the variational principle and infimal convolution. Hence, the aim of this paper is to give a study on a relationship between Ekeland's variational principle and infimal convolution.

By help of graphic interpretation of infimal convolution, Ekeland's variational principle can be regarded as an algorithm similar to the proximal point algorithm for a problem of minimizing a lower semicontinuous proper convex function on a Hilbert space; see [7, 9, 11]. These algorithms are included in a more general optimization algorithm related to infimal convolution; we will call the algorithm "Ekeland's algorithm."

In this paper, we present that Ekeland's variational principle is an algorithm finding a solution to be the minimizer of $f(x)$ subject to elements x at which the infimal convolution $f \triangledown d$ of the objective function f and the metric function d is exact, that is, there is $x = x_1 + x_2$ such that

$$f(x_1) + d(x_2) = \inf_{\substack{u,v\in X\\ u+v=x}} f(u) + d(v) =: f \triangledown d(x).$$

The presentation consists of five elementary results and three interesting theorems. First theorem shows that the exactness of infimal convolution is necessary and sufficient for the Ekeland's algorithm to be feasible. Second theorem is an extended version of Ekeland's variational principle in a topological vector space under considerable tight conditions. Third theorem is an extended version of the variational principle in a complete metric space.

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2. GRAPHIC INTERPRETATION OF INFIMAL CONVOLUTION

Throughout the paper, let X be a set, $f : X \to \boldsymbol{R} \cup \{+\infty\}$ and $g : X \times X \to \boldsymbol{R} \cup \{+\infty\}$ positive functions. The domain of f is the set

$$\textbf{dom}\, f := \{x \in X \mid f(x) < +\infty\}.$$

A function $f : X \to \boldsymbol{R} \cup \{+\infty\}$ is said to be strict if its domain is non-empty. The epigraph of f and the hypograph of f are the subsets of $X \times \boldsymbol{R}$ defined by

$$\textbf{epi}\, f := \{(x, \mu) \in X \times \boldsymbol{R} \mid f(x) \leq \mu\},$$

$$\textbf{hyp}\, f := \{(x, \mu) \in X \times \boldsymbol{R} \mid f(x) \geq \mu\}.$$

The epigraph of f is non-empty if and only if f is strict. For the existence of a solution to a minimization problem, compactness plays a crucial role. However, simply with the condition that set over which an objective function is minimized is complete, Ekeland's variational principle gives us an existence result for an approximate minimization problem. This principle has verified to be a major tool in nonlinear analysis with a wide range of applications, e.g., convex analysis, optimization theory, and geometry of Banach spaces. To begin with, we recall the principle, which is illustrated in Figure 2.1 graphically:

> Let (X, d) be a complete metric space, and let $f : X \to \boldsymbol{R} \cup \{+\infty\}$ be strict, positive, lower semicontinuous (l.s.c., in short). Then, for $x_0 \in \textbf{dom}\, f$ and $\varepsilon > 0$ there exists $\bar{x} \in X$ such that
>
> $$\begin{array}{rcll} f(\bar{x}) & \leq & f(x_0) - \varepsilon d(x_0, \bar{x}), & \\ f(x) & > & f(\bar{x}) - \varepsilon d(\bar{x}, x) & \forall x \in X, x \neq \bar{x}. \end{array}$$

The graphic interpretation for the principle gives us a motivation for the following observation. We consider the maximal quantity of lifting the graph of $-g(x, \cdot)$ not beyond the graph of f:

$$G(f, g; x) := \sup\{\mu \mid f(y) \geq -g(x, y) + \mu, \forall y \in X\} \qquad x \in X.$$

In the case of Ekeland's variational principle, the function g is first-order type of the form $g = \varepsilon d(y, x)$. This is illustrated in Figure 2.2. Then, the function G has the following properties.

Lemma 2.1. *The following statements hold:*

(i) $0 \leq G(f, g; x) \leq f(x)$ *if* $f, g \geq 0$ *and* $g(x, x) = 0$;

(ii) $\textbf{dom}\, f \subset \textbf{dom}\, G(f, g; \cdot)$ *if* $f, g \geq 0$ *and* $g(x, x) = 0$;

(iii) $G(f, g; x) = f \triangledown h(x)$ *if* $g(x, y) = h(x - y)$ *where* $h : X \to \boldsymbol{R} \cup \{+\infty\}$;

(iv) $G(f, g; x) = \sup\{\mu \mid \textbf{epi}\, f \cap (\textbf{hyp}\,(-h(\cdot - x)) + (0, \mu)) = \emptyset\}$ *if* $g(x, y) = h(x - y) = h(y - x)$ *and* $\textbf{dom}\, h(\cdot - x) \neq \emptyset$ *where* $h : X \to \boldsymbol{R} \cup \{+\infty\}$;

(v) $\textbf{epi}\, f \cap \textbf{int}\, (\textbf{hyp}\,(-h(\cdot - x)) + (0, G(f, g; x))) = \emptyset$ *if* $g(x, y) = h(x - y) = h(y - x)$ *and* $\textbf{int dom}\, h(\cdot - x) \neq \emptyset$ *where* $h : X \to \boldsymbol{R} \cup \{+\infty\}$.

In particular, (iii) of Lemma 2.1 is a key property in this paper.

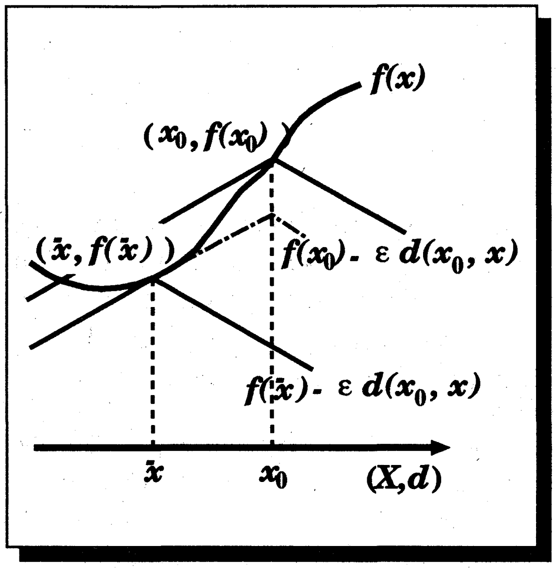

Figure 2.1: **Ekeland's variational principle**

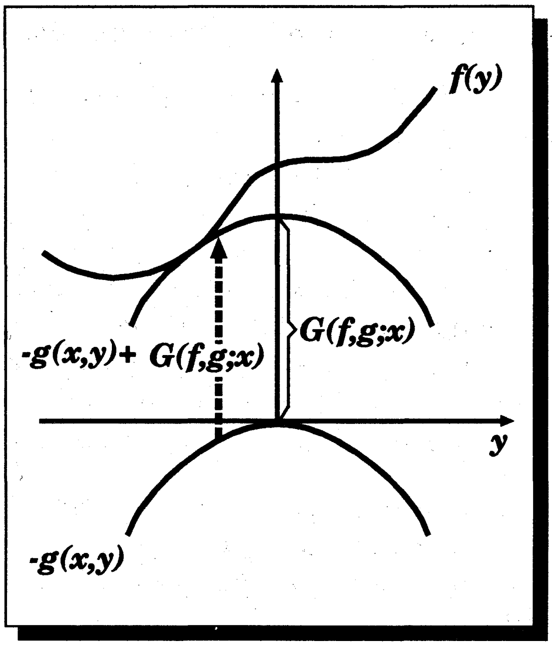

Figure 2.2: **the maximal quantity of lifting the graph of $-g(x,\cdot)$ not beyond the graph of f**

3. RELATIONSHIP BETWEEN INFIMAL CONVOLUTION AND EKELAND'S ALGORITHM

Based on the results in Lemma 2.1, we observe the relationship between the exactness of infimal convolution and the feasibleness of Ekeland's algorithm.

Theorem 3.1. *Let X be a vector space, $f : X \to \boldsymbol{R} \cup \{+\infty\}$ a strict function. Assume that $g : X \times X \to \boldsymbol{R} \cup \{+\infty\}$ is given by $g(x,y) = h(x-y)$, where $h : X \to \boldsymbol{R} \cup \{+\infty\}$ satisfies $h(-x) = h(x)$, $h(0) = 0$, $h(x) \geq 0$ for all $x \in X$. Given $x_0 \in$ **dom** f, we define the set*

$$\Gamma(x_0) := \{x \in X \mid f(x) = -h(x - x_0) + G(f, h; x_0)\}.$$

Then, the infimal convolution $f \triangledown h$ is exact at $x_0 = \bar{x} + (x_0 - \bar{x})$, that is,

$$f \triangledown h(x_0) = f(\bar{x}) + h(x_0 - \bar{x})$$

if and only if $\bar{x} \in \Gamma(x_0)$, which means that the Ekeland's algorithm is feasible at x_0 and that $f(\bar{x}) \leq f(x_0) - h(\bar{x} - x_0)$.

Proof. Let the infimal convolution $f \triangledown h$ be exact at $x_0 = \bar{x} + (x_0 - \bar{x}_0)$. By (iii) of Lemma 1, we have

$$G(f, h; x_0) = f \triangledown h(x_0) = f(\bar{x}) + h(x_0 - \bar{x}),$$

$$f(\bar{x}) = -h(\bar{x} - x_0) + G(f, h; x_0),$$

and hence

$$\bar{x} \in \Gamma(x_0).$$

The converse is also easy to prove. Moreover, by (i) of Lemma 1, we have

$$f(\bar{x}) + h(x_0 - \bar{x}) = G(f, h; x_0) \leq f(x_0),$$

$$f(\bar{x}) \leq f(x_0) - h(x_0 - \bar{x}).$$

∎

This theorem shows that the feasibleness of the algorithm is equivalent to the exactness of the infimal convolution of the functions f and h. Hence, we will call this algorithm $x \mapsto \Gamma(x)$, which is a set-valued mapping, Ekeland's algorithm. It is, however, difficult to check the exactness of infimal convolution in almost all general cases. In the finite dimensional case (or another special case), we have the following result with a simple assumption.

Theorem 3.2. *Let X be a topological vector space, $f : X \to \boldsymbol{R} \cup \{+\infty\}$ a strict positive lower semicontinuous function, and $x_0 \in$ **dom** f. Assume that $h : X \to \boldsymbol{R} \cup \{+\infty\}$ is a lower semicontinuous function such that $h(-x) = h(x)$, $h(0) = 0$, $h(x) \geq 0$ for all $x \in X$. Either if $\dim X < +\infty$ or if **epi** $h \cap \{(x, \mu) \mid \mu \leq \alpha\}$ is a compact set for each $\alpha > 0$, then there exists $\bar{x} \in X$ such that $\bar{x} \in \Gamma(x_0)$.*

Proof. By the assumption, **epi** f is closed and

$$\textbf{hyp } (-h(\cdot - x_0) + (0, G(f, g; x_0))) \cap \{(x, \mu) \mid \mu \geq 0, x \in X\}$$

is compact. Suppose to the contrary that $\Gamma(x_0) = \emptyset$, then there is an (open balanced absorbing) nbd V of $(0, 0) \in X \times \boldsymbol{R}$ such that

$$(\textbf{epi } f + V) \cap (K + V) = \emptyset.$$

Since V is absorbing, there is $r > 0$ such that $(0, r) \in V$, and hence

$$\textbf{epi } f \cap \{\textbf{hyp } (-h(\cdot - x_0)) + (0, G(f, g; x_0) + r)\} = \emptyset,$$

which implies that

$$G(f, g; x_0) \leq f \triangledown h(x_0) - r < f \triangledown h(x_0).$$

We have a contradiction to (iii) of Lemma 2.1. ∎

This theorem is an extended version of Ekeland's variational principle to a topological vector space, not necessary a complete metric space.

Finally, we present another extended version of Ekeland's variational principle in a complete metric space.

Theorem 3.3. *Let (X, d) be a complete metric space, $f : X \to \boldsymbol{R} \cup \{+\infty\}$ a strict positive lower semicontinuous function. Assume that $g : X \times X \to \boldsymbol{R} \cup \{+\infty\}$ is lower semicontinuous in the second argument satisfying that*

(i) $g(x, x) = 0$ *for all* $x \in X$;

(ii) $g(x, y) \leq g(x, z) + g(z, y)$ *for all* $x, y, z \in X$;

(iii) $g(x, y) > g(x, z) > 0$ *if* $d(x, y) > d(x, z) > 0$ *for all* $x, y, z \in X$.

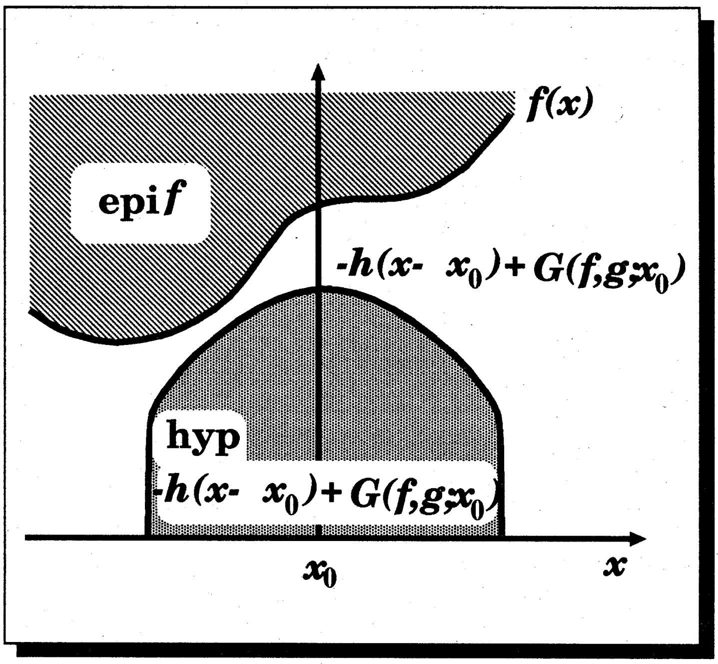

Figure 3.3: **an illustration for Theorem 3.2**

Then, for $x_0 \in \mathbf{dom}\, f \cap \mathbf{dom}\, g$ *there exists* $\bar{x} \in X$ *such that*

$$f(\bar{x}) \leq f(x_0) - g(x_0, \bar{x}),$$

$$f(x) > f(\bar{x}) - g(\bar{x}, x) \text{ for all } x \in X, x \neq \bar{x}.$$

Proof. The method of proof is the same as that of [8]. We briefly give the outline of the proof. To each x_n we adjoin the closed set

$$S_n := \{x \in X \mid f(x) \leq f(x_n) - g(x_n, x)\}$$

and define

$$\gamma_n := \inf_{x \in S_n} f(x) - f(x_n).$$

Since $g(x_n, x_n) = 0$, we have $x_n \in S_n$, which shows that $S_n \neq \emptyset$, and that $\gamma_n \leq 0$. Then, we can choose $x_n \in S_{n-1}$ such that

$$f(x_n) - f(x_{n-1}) \leq \gamma_{n-1} + \frac{1}{n},$$

and we observe $\gamma_n \geq -\frac{1}{n}$ and the diameter of S_n tends to zero. Hence, the sequence $\{x_n\}$ is Cauchy and tends to a limiting point $\bar{x} \in X$, where

$$\bigcap_{n=0}^{\infty} S_n = \{\bar{x}\}.$$

From $\bar{x} \in S_0$, we have $f(\bar{x}) \leq f(x_0) - g(x_0, \bar{x})$. Next, we assume to the contrary that there is $x^* \in X$, $x^* \neq \bar{x}$, such that $f(\bar{x}) < f(\bar{x}) - g(\bar{x}, x^*)$. Since $\bar{x} \in \bigcap_{n=0}^{\infty} S_n$, we have

$$\begin{aligned} f(\bar{x}) &< f(\bar{x}) - g(\bar{x}, x^*) \\ &\leq f(x_n) - g(x_n, \bar{x}) - g(\bar{x}, x^*) \\ &\leq f(x_n) - g(x_n, x^*) \quad \text{for all } n, \end{aligned}$$

which shows that $x^* \in \bigcap_{n=0}^{\infty} S_n$. This is a contradiction to $x^* \neq \bar{x}$. ∎

4. CONCLUSIONS

In this paper, we have presented a relationship between Ekeland's variational principle and infimal convolution, and proposed Ekeland's algorithm to obtain a point of **epi** f contacting with the lifted **hyp** g, which is based on the fact that the feasibleness of the algorithm is equivalent to the exactness of the infimal convolution of the functions f and h. Then, we have proved in Theorem 3.1 that Ekeland's variational principle is an algorithm finding a solution to be the minimizer of $f(x)$ subject to elements x at which the infimal convolution $f \triangledown d$ of the objective function f and the metric function d is exact, that is, there is $x = x_1 + x_2$ such that

$$f(x_1) + d(x_2) = \inf_{\substack{u,v \in X \\ u+v=x}} f(u) + d(v) =: f \triangledown d(x).$$

We have also given extended Versions of Ekeland's variational principle. One is Theorem 3.2, which is an extended version of Ekeland's variational principle to a topological vector space, not necessary a complete metric space. The other is Theorem 3.3, which is an extended version of Ekeland's variational principle in a complete metric space.

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