数理解析研究所講究録
第 897 巻 1995 年 133-142

A Fixed Point Theorem for Noncommutative Families of Nonexpansive Mappings in Banach spaces

by

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Abstract

Let C be a nonempty weakly compact convex subset of a Banach space which has normal structure and let S be a semitopological semigroup such that $RUC(S)$ has a left invariant mean. Then we prove a fixed point theorem for a continuous representation of S as nonexpansive mappings on C.

1991 Mathematics Subject Classification. Primary 47H10.
Key words and phrases. Fixed point, nonexpansive mapping, invariant mean.

1 Introduction.

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$, the mappings $s \to sa$ and $s \to as$ from S into S are continuous and let $RUC(S)$ be the space of bounded right uniformly continuous functions on S. Let C be a nonempty subset of a Banach space and let $\mathcal{S} = \{T_t : t \in S\}$ be a family of self-maps of C. $\mathcal{S}$ is said to be a continuous representation of S as nonexpansive mappings on C if the following conditions are satisfied :

(1) $T_{st}x = T_sT_tx$ for all $t, s \in S$ and $x \in C$;

(2) for each $x \in C$, the mapping $s \to T_sx$ from S into C is contiunous.

Let $F(\mathcal{S})$ denote the set of common fixed points of T_s, $s \in S$. Fixed point theorems for noncommutative families of nonexpansive mappings on C have been investigated by several authors ; see, for example, Bartoszek[1], Holmes-Lau[2,3], Lau[4,5,6], Lau-Takahashi[7,8], Lim[9,10], Mitchell[11,12], Takahashi[13,14,15,16], Takahashi-Jeong[17] and others. Among these, Lim[9] proved that if S is left reversible (i.e., any two closed right ideals in S have non-void intersection) and C is weakly compact, convex, and has normal structure, then $\mathcal{S}$ has a common fixed point in C.

In this paper, we prove a fixed point theorem for a continuous representation of S as nonexpansive mappings on C in the case of which $RUC(S)$

has a left invariant mean and C is weakly compact, convex, and has normal structure. It is well known that left reversibility and existence of a left invariant mean on $RUC(S)$ do not imply each other.

2 Fixed point theorem.

Let S be a set and $m(S)$ be the Banach space of all bounded real-valued functions on S with the supremum norm. Let X be a subspace of $m(S)$ containing constants. Then $\mu \in X^*$ is called a mean on X if $\|\mu\| = \mu(1) = 1$. Let $\mu \in X^*$ be a mean on X and $f \in X$. Then we denote by $\mu(f)$ the value of μ at the function f. According to time and circumstances, we write $\mu_t(f(t))$ the value $\mu(f)$. As is well known, $\mu \in X^*$ is a mean on X if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s)$$

for every $f \in X$. If S is a semigroup, $a \in S$, and $f \in m(S)$, define $(\ell_a f)(t) = f(at)$ and $(r_a f)(t) = f(ta)$, $t \in S$. If $\ell_a(X) \subseteq X$ for all $a \in S$, then a mean μ on X is left invariant if $\mu(\ell_a f) = \mu(f)$ for all $a \in S$ and $f \in X$. Let S be a semitopological semigroup. Let $C(S)$ be the Banach space of bounded continuous real-valued functions on S. Let $RUC(S)$ denote the space of bounded right uniformly continuous functions on S, i.e., all $f \in C(S)$ such that the mapping $s \to r_s f$ of S into $C(S)$ is continuous. Then $RUC(S)$ is a closed subalgebra of $C(S)$ containing constants and invariant under left and right translations (see [12] for details). A closed convex subset C of a Banach space is said to have normal structure if for each closed bounded convex subset K of C, which contains at least two points, there exists an element of K which is not a diametral point of K. It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure. Lim[9] also proved the following.

Lemma[9]. A closed convex subset C of a Banach space has normal structure if and only if it does not contain a sequence $\{x_n\}$ such that for some $c > 0$, $\|x_n - x_m\| \leq c$, $\|x_{n+1} - \overline{x_n}\| \geq c - \frac{1}{n^2}$ for all $n \geq 1$, $m \geq 1$, where $\overline{x_n} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

Now we can prove a fixed point theorem for noncommutative families of nonexpansive mappings in Banach spaces.

Theorem. Let S be a semitopological semigroup, let D be a weakly compact subset of a Banach space B which has normal structure and let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as nonexpansive mappings on D. Suppose $RUC(S)$ has a left invariant mean. Then $\mathcal{S}$ has a common fixed point in D.

Proof. We first prove that for any $x \in D$ and $y \in B$, a function h defined by $h(t) = \|T_t x - y\|$ for all $t \in S$ is in $RUC(S)$. In fact, we have, for

$s, u \in S$,

$$\begin{aligned}\|r_s h - r_u h\| &= \sup_{t\in S} |(r_s h)(t) - (r_u h)(t)| = \sup_{t\in S} |h(ts) - h(tu)| \\ &= \sup_{t\in S} |\|T_{ts}x - y\| - \|T_{tu}x - y\|| \le \sup_{t\in S} \|T_{ts}x - T_{tu}x\| \\ &\le \|T_s x - T_u x\|.\end{aligned}$$

Let

$$E = \{K \subset D : K \text{ is nonempty, closed, convex, and } T_s\text{-invariant } \}.$$

Then by Zorn's Lemma, there exists a minimal element C of E. Let $\delta(C) > 0$ and let μ be a left invariant mean. Then, for any $x \in C$,

$$A_x = \{z \in C : \mu_t \|T_t x - z\| = \min_{y\in C} \mu_t \|T_t x - y\|\}$$

is nonempty, closed, convex, and T_s-invariant (see [8,13] for details). So, we have $A_x = C$ from minimality of C. Since μ is a mean, there exists a net of finite means λ_α such that $\lambda_\alpha \longrightarrow^{w^*} \mu$. Let $x_0 \in C$, $\epsilon > 0$, and $x_1, x_2, \cdots, x_n \in C$. Since $A_{x_0} = C$, there exists α_0 such that

$$(\mu_{\alpha_0})_t \|T_t x_0 - x_i\| \le r + \epsilon, \quad \forall i = 1, 2, \cdots, n,$$

where $r = \min_{y\in C} \mu_t \|T_t x_0 - y\|$. That is, there exists $z = \sum_{j=1}^{n_{\alpha_0}} \lambda_j T_{s_j} x_0$ with $\lambda_1, \cdots,$ $\lambda_{n_{\alpha_0}} \ge 0$ and $\sum_{j=1}^{n_{\alpha_0}} \lambda_j = 1$ such that

$$\|z - x_i\| \le \sum_{j=1}^{n_{\alpha_0}} \lambda_j \|T_{s_j} x_0 - x_i\| \le r + \epsilon, \quad \forall i = 1, 2, \cdots, n. \tag{1}$$

Let $C_{y,\epsilon} = \{z \in C : \|z - y\| \leq r + \epsilon\}$ for each $y \in C$. Then by (1),

$$\{C_{y,\epsilon} : y \in C\}$$

has finite intersection property. Since C is weakly compact, there is $z_0 \in C$ such that $\|z_0 - y\| \leq r + \epsilon$ for every $y \in C$. Since $\{T_t x_0\} \subset C$, we have $sup_{t \in S}\|z_0 - T_t x_0\| \leq \sup_{y \in C} \|z_0 - y\| \leq r + \epsilon$. Since

$$r = \mu_t \|T_t x_0 - z_0\| \leq \sup_t \|T_t x_0 - z_0\| \leq \sup_{y \in C} \|z_0 - y\| \leq r + \epsilon$$

and

$$r = \mu_t \|T_t x_0 - x\| \leq \sup_t \|T_t x_0 - x\| \leq \sup_{y \in C} \|y - x\|, \quad \forall x \in C,$$

we have

$$r \leq \inf_{x \in C} \sup_t \|T_t x_0 - x\| \leq \inf_{x \in C} \sup_{y \in C} \|y - x\| \leq r + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$r = \mu_t \|T_t x_0 - x\| = \inf_{z \in C} \sup_{y \in C} \|y - z\|, \quad \forall x \in C. \tag{2}$$

Since $x_0 \in C$ is arbitrary, for any $x, z \in C$, we have

$$r = \mu_t \|T_t x - z\| = \inf_{u \in C} \sup_{t \in S} \|T_t x - u\| = \inf_{u \in C} \sup_{y \in C} \|y - u\|.$$

So, let

$$A_0 = \{z \in C : \sup_{t \in S} \|z - T_t x\| \leq r, \quad \forall x \in C\}.$$

By (2), since there exists $z_0 \in C$ such that

$$\sup_{y\in C} \|y - z_0\| = r,$$

we have that A_0 is nonempty. Let $z_0 \in A_0$ and $s \in S$. Then putting

$$A_s = \{z \in C : \sup_{t\in S} \|T_{st}x - z\| \le r, \ \ \forall x \in C\},$$

we have z_0, $T_s z_0 \in A_s$. Further, for any $x \in C$,

$$\begin{aligned} r = \mu_t\|T_t x - z_0\| &= \mu_t\|T_{st}x - z_0\| \le \sup_{t\in S} \|T_{st}x - z_0\| \\ &\le \sup_{t\in S} \|T_t x - z_0\| \le r. \end{aligned}$$

and

$$\begin{aligned} r = \mu_t\|T_t x - T_s z_0\| &= \mu_t\|T_{st}x - T_s z_0\| \le \sup_{t\in S} \|T_{st}x - T_s z_0\| \\ &\le \sup_{t\in S} \|T_t x - z_0\| \le r. \end{aligned}$$

For using Lim's Lemma, fix $z_0 \in A_0$. Then since $r = \mu_t\|T_t z_0 - z_0\|$, there exists $s_1 \in S$ such that $\|T_{s_1} z_0 - z_0\| \ge r - 1$. Since $z_0, T_{s_1} z_0 \in A_{s_1}$ and A_{s_1} is convex,

$$\overline{x_2} = \frac{1}{2} z_0 + \frac{1}{2} T_{s_1} z_0 \in A_{s_1}.$$

Let $x_1 = z_0$ and $x_2 = T_{s_1} z_0$. Since $r = \mu_t\|T_t z_0 - \overline{x_2}\| = \mu_t\|T_{s_1 t} z_0 - \overline{x_2}\|$, there exists $s_2 \in S$ such that $\|T_{s_1 s_2} z_0 - \overline{x_2}\| \ge r - \frac{1}{2^2}$. So, let $x_3 = T_{s_1 s_2} z_0$. Then, we have

$$\|x_1 - x_2\| = \|z_0 - T_{s_1} z_0\| \le \sup_{t\in S} \|z_0 - T_t z_0\| = r,$$

$$\|x_2 - x_3\| = \|T_{s_1}z_0 - T_{s_1s_2}z_0\| \le \|z_0 - T_{s_1}z_0\| \le r,$$

and

$$\|x_3 - x_1\| = \|T_{s_1s_2}z_0 - z_0\| \le \sup_{t\in S}\|T_t z_0 - z_0\| = r.$$

Similarly, let

$$\overline{x_3} = \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3.$$

Then, $r = \mu_t\|T_t z_0 - \overline{x_3}\| = \mu_t\|T_{s_1s_2t}z_0 - \overline{x_3}\|$, there exists $s_3 \in S$ such that $\|T_{s_1s_2s_3}z_0 - \overline{x_3}\| \ge r - \frac{1}{3^2}$. So, let $x_4 = T_{s_1s_2s_3}z_0$. Then, we have

$$\|x_4 - x_1\| = \|T_{s_1s_2s_3}z_0 - z_0\| \le \sup_{t\in S}\|T_t z_0 - z_0\| = r,$$

$$\|x_4 - x_2\| = \|T_{s_1s_2s_3}z_0 - T_{s_1}z_0\| \le \|T_{s_2s_3}z_0 - z_0\| \le \sup_{t\in S}\|T_t z_0 - z_0\| = r,$$

and

$$\|x_4 - x_3\| = \|T_{s_1s_2s_3}z_0 - T_{s_1s_2}z_0\| \le \|T_{s_3}z_0 - z_0\| \le \sup_{t\in S}\|T_t z_0 - z_0\| = r.$$

By mathematical induction, let $x_5 = T_{s_1s_2s_3s_4}z_0, x_6 = T_{s_1s_2s_3s_4s_5}z_0, \cdots$. Then we have

$$\|x_n - x_m\| \le r, \quad \forall n, m \quad \text{and} \quad \|x_{n+1} - \overline{x_n}\| \ge r - \frac{1}{n^2}.$$

Using Lim's Lemma, C has not normal structure. This is a contradiction.

References

[1] W.Bartoszek, Nonexpansive actions of topological semigroups on strictly convex Banach space and fixed points, Proc.Amer.Math.Soc., 104(1988), 809-811.

[2] R.H.Holmes and A.T.Lau, Asymptotically non-expansive actions of topological semigroups and fixed points, Bull. London Math. Soc., 3(1971), 343-347.

[3] R.H.Holmes and A.T.Lau, Nonexpansive actions of topological semigroups and fixed points, J. London Math. Soc., 5(1972), 330-336.

[4] A.T.Lau, Invariant means on almost periodic functions and fixed point properties, Rocky Mountain J. Math., 3(1973), 69-76.

[5] A.T.Lau, Some fixed point theorems and their applications to W*-algebras, in Fixed Point Theory and Its Applications (S. Swaminathan Ed.), Academic Press, Orlando, FL, (1976), 121-129.

[6] A.T.Lau, Amenability and fixed point property for semigroup of non-expansive mappings, in Fixed Point Theory and Applications (M.A.Théra and J.B.Baillon Eds.), Pitman Research Notes in Mathematics Series ♯252 (1991), 303-313.

[7] A.T.Lau and W.Takahashi, Weak convergence and non-linear ergodic theorems for reversible semigroups of nonexpansive mappings, Pacific J. Math., 126(1987), 277-294.

[8] A.T.Lau and W.Takahashi, Invariant means and semigroups of nonexpansive mappings on uniformly convex Banach spaces, J. Math. Anal. Appl., 153(1990), 497-505.

[9] T.C.Lim, A fixed point theorem for families of nonexpansive mappings, Pacific J. Math., 53(1974), 484-493.

[10] T.C.Lim, Asymptotic centers and nonexpansive mappings in conjugate Banach spaces, Pacific J. Math., 90(1980), 135-143.

[11] T.Mitchell, Fixed points of reversible semigroups of non-expansive mappings, Kōdai Math. Sem. Rep., 2(1970), 322-323.

[12] T.Mitchell, Topological semigroups and fixed points, Illinois J. Math., 14(1970), 630-641.

[13] W.Takahashi, Fixed point theorem for amenable semigroups of non-expansive mappings, Kōdai Math. Sem. Rep., 21(1969), 383-386.

[14] W.Takahashi, A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space, Proc. Amer. Math. Soc., 81(1981), 253-256.

[15] W.Takahashi, Fixed point theorems for families of nonexpansive mappings on unbounded sets, J. Math. Soc. Japan, 36(1984), 543-553.

[16] W.Takahashi, Fixed point theorem and nonlinear ergodic theorem for nonexpansive semigroups without convexity, Can. J. Math., 35(1992), 1-8.

[17] W.Takahashi and D.H.Jeong, Fixed point theorem for nonexpansive semigroups on Banach space, Proc. Amer. Math. Soc., 122(1994), 1175-1179.