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DISCONTINUITY OF SOLUTIONS OF PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS WITH TIME DELAY IN HILBERT SPACE

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0. Introduction and Theorem.

In this paper we consider the following integro-differential equation with time delay in a real Hilbert space H:

$$(0.1) \qquad \frac{d}{dt}u(t) + Au(t) + A_1 u(t-h) + \int_{-h}^{0} a(-s)A_2 u(t+s)ds = f(t)$$

$$u(0) = x, \quad u(s) = y(s) \qquad -h \leq s < 0.$$

Here, A is a positive definite self-adjoint operator and A_1 , A_2 are closed linear operators with domains containg that of A. The notations h and N denote a fixed positve number and a large natural number respectively. Let $a(\cdot)$ is a real valued function belonging to $C^3([0,h])$.

The equations of the type (0.1) were investigated by G.Di Blasio, K.Kunisch and E.Sinestrari [2], S.Nakagiri [4], H.Tanabe [6] and D.G.Park and S.Y.Kim [5]. Particulary, G.Di Blasio, K.Kunisch and E.Sinestrari [2] showed the existence and uniquness of a solution for $f \in L^2(0,T;H)$, $Ay \in L^2(-h,0;H)$ and $x \in (D(A),H)_{1/2,2}$ where $(D(A),H)_{1/2,2}$ is a interpolation space.

Since the equation (0.1) is of parabolic type, we want x to be an arbitrary element of H. Then the integral in (0.1) exists only in the improper sence no matter what nice functions f and Ay may be. Hence, it would be considered natural to investigate our problem under the following hypothesis:

$$f \in \cap_{\delta>0} L^2(\delta, T; H) \quad \text{and} \quad Ay \in \cap_{\delta>0} L^2(-h+\delta, 0; H),$$

$$f(t) \text{ and } Ay(t-h) \text{ are improperly integrable at} t=0.$$

For the sake of simplicity we put

$$L^2_{loc}((0,T];H) = \cap_{\delta>0} L^2(\delta, T; H).$$

We first shall state the deinition of a weak solution of (0.1).

DEFINITION. We say that a function u definited on $[-h, T]$ is a weak solution of the equation (0.1) if the following four conditions satisfied:(see Definition 1.1 in [3])

1) $u \in L^2_{loc}((nh,(n+1)h]; D(A)) \cap W^{1,2}_{loc}((nh,(n+1)h]H) \cap C([0, Nh]; D(A^{-\alpha}))$
   for $n = 0, 1, 2, \cdot, \cdot, , N-1$ and any $\alpha > 0$.
2) $lim_{t\to 0} A^{-\alpha} u(t) = A^{-\alpha} x$
   for any $\alpha > 0$ and $u(s) = y(s)$ for $-h \leq s < 0$.
3) $Au(\cdot + nh) \in L^2_{loc}((0,h];H)$ and $A^{1-\alpha} u(\cdot + nh)$ is improper integrable at $t = 0$.
4) The function u satisfies the equation (0.1) for a.e t.

In Theorem 1 in [3] we showed the existence and uniqueness of a weak solution for which $A^{-\alpha} u$ is continuous in [0,T] for an arbitray positive number α but this solution is not alway in $C([0,T];H)$.

As the notations we put

$$F_{-1} = \{g \in L^2_{loc}((0,h] : H); \text{ there exists} \lim_{\epsilon \searrow 0} \int_\epsilon g(s) ds.\},$$

$$F_m = \{g \in F_{m-1}; \lim_{t \searrow 0} \int_{t/2}^{t} (t-s)^m A_1^m S(t-s) g(s) ds = 0\}$$

where $S(\cdot)$ is an analitic semigroup of the positive defined self-ajoint operator A and $m = 1, 2, \cdot\cdot, N-1$.

In Proposition 6.9 of [3] we also showed the following resultant.

Let f belong to $F_{-1} \cap L^2_{loc}((0, Nh] : H)$ and m is a nonnegative integer such that $0 \leq m \leq N-1$. Then following two conditions are equivalent.

1) A weak solution of (0.1) is continuous on $[0, mh]$, but at $t = mh$ this solution is discontinuous.

2) $f - A_1 y(\cdot - h) \in F_{m-1}$, but $f - A_1 y(\cdot - h) \notin F_m$.

In [3] we could not show that F_m is a proper subset in F_{m-1}. The object in this paper is to show that F_m is a proper subset in F_{m-1} (i.e there exists a inhomogeneous function f and a initial data function y such that the solution of (0.1) is continuous on [0,mh], but at $t = mh$ this solution is discontinuous on H.)

Throughout this paper we assume

$A-1)$ $A = A_1 = A_2$,

$A-2)$ the operator A holds eigenvalues $\{\lambda_q\}_{q=1}^{\infty}$ such that

$$\lambda_q = Cq^\alpha + o(q^\alpha), \qquad \lambda_q \leq \lambda_{q+1} \tag{0.2}$$

where α and C are some positive numnbers. We denote normal eigenfuctions of eigenvalues λ_q by φ_j.

THEOREM *Under the assumptions A-1) and A-2) there exist a inhomogeneous function f and the initial valued function y such that the weak solution of (0.1) is continuous on $[0, mh]$, but at $t = mh$ it is discontinuous.*

1. Properties of eigenvalues.

We denote 10^{-1} by ϵ_0.

LEMMA 1. *Let ϵ_0 be a small positive number and t_0 be sufficiently small positive number. Then there exists a eigenvalue λ_q such that*

$$(1.1).\qquad 1-\epsilon_0 < t\lambda_q < 1+\epsilon_0 \qquad \text{for any} \quad t : 0 < t < t_0.$$

Proof. We suppose that there exists a small positive number t_0 such that

$$t\lambda_q \leq 1-\epsilon_0 \quad \text{or} \quad t\lambda_q \geq 1+\epsilon_0 \qquad \text{for any natural number} \quad q.$$

We put $p = \max_q\{q : \lambda_q \leq (1-\epsilon_0)/t\}$ and $r = \min_q\{q : \lambda_q \geq (1+\epsilon_0)/t$. If t_0 is sufficiently small, p and r are sufficiently large natual number and $p+1 = r$. From the assumption A-2) and (1.1) we get

$$Cp^\alpha + o(p^\alpha) \leq (1-\epsilon_0)/t \quad \textit{and} \quad C(p+1)^\alpha + o((p+1)^\alpha) \geq (1+\epsilon_0)/t.$$

Then it follows

$$(1+\epsilon_0)(C(p+1)^\alpha + o((p+1)^\alpha))^{-1} \leq t \leq (1-\epsilon_0)(Cp^\alpha + o(p^\alpha))^{-1}.$$

Since p is sufficiently large natual number we obtain that the above inequalities are contadiction. Thus the proof is complte.

Let θ and N be $1/3 - 4/(3N)$ and 10^3 respectively.

We choose a sequence $\{t_n\}$ such that $t_1 = t_0/2$ and $0 < t_{n+1} < t_n\theta^n/2$ for any $n = 1, 2, 3, 4, \cdots$.
where t_0 is of lemma 1

LEMMA 2. *Let j and n be natural number such that $0 < j \leq n$. Thus there exists a natural number $\ell(n,j)$ such that*

$$1-\epsilon_0 < (\theta^j t_n)\lambda_{\ell(n,j)} < 1+\epsilon_0,$$

$$\textit{and if } (n_1, j_1) \neq (n_2, j_2) \quad \textit{then } \lambda_{\ell(n_1,j_1)} \neq \lambda_{\ell(n_2,j_2)}.$$

where $\epsilon_0 = 10^{-1}$.

Proof. Since t_0 is sufficiently small positive number, from Lemma 1, we see that there exists λ_ℓ. Next we shall show the eigenvalue is unique. Suppose $(n_1, j_1) \neq (n_2, j_2)$ and $n_1 \geq n_2$. Then if $n_1 > n_2$ it follows $t_{n_2}\theta^{j_2} > 2t_{n_1}\theta^{j_1}$. If $n_1 = n_2$ and $j_1 > j_2$ it also follows $t_{n_2}\theta^{j_2} > 2t_{n_1}\theta^{j_1}$. From (1.1) and the above inequalities we have

$$\lambda_{\ell(n_2,j_2)} < (1+\epsilon_0)(t_{n_2}\theta^{j_2})^{-1} < (1+\epsilon_0)2^{-1}(t_{n_1}\theta^{j_1})^{-1} < (1+\epsilon_0)(1-\epsilon_0)^{-1}2^{-1}\lambda_{\ell(n_1,j_1)}.$$

Thus it follows $\lambda_{\ell(n_2,j_2)} < \lambda_{\ell(n_1,j_1)}$.

2. Constitution of functions.

We shall constitute our aim's function which satisfies the following conditions:

$$f \in F_{m-1} \cap L^2_{loc}((0,h]; H) \quad but \quad \notin F_m.$$

For the sake of simplicity we suppose $h = 1$.

We first take a sequence $\{x_{n,j}\}$ such that

$$x_{n,0} = 2^{-1}t_n \quad \text{and} \quad x_{n,j} = x_{n,j-1} + (1 + 2/N)\theta^{j-1}t_n/3$$

where $n = 1, 2, \cdot, \cdot, \cdot$ and $j = 1, 2, \cdot, \cdot \leq n$.

REMARK 1. Since $\sum_{j=1}^{n}(1 + 2/N)\theta^{j-1}/3 \leq 1/2$ it follows $t_n/2 \leq x_{n,j} < t_n$ where $j = 0, 1, 2, \cdot\cdot, n$.

For the sake of the simplicity we put $\gamma_{n,j} = \theta^j t_n/(3N)$, and $\Gamma_{n,j} = (1 + 1/N)\theta^j t_n/3$.

Let χ_1 and χ_2 be functions such that

1) $\chi_1, \chi_2 \in C^\infty([0,1])$,

2) Supp $\chi_1 \subset [2^{-1}, 1]$ and Supp $\chi_2 \subset [0, 2^{-1}]$,

3) $\chi_1(\cdot) = 1$ on $[2/3, 1]$ and $\chi_2(\cdot) = 1$ on $[0, 1/3]$.

We denote $\chi_1((t - x_{n,j})/\gamma_{n,j})$ and $\chi_2((t - x_{n,j} - \Gamma_{n,j})/\gamma_{n,j})$ by $\chi_{1,n,j}(t)$ and $\chi_{2,n,j}(t)$ respectively.

Let p be an arbitrary natural number. We define a function $f^p_{n,j}(t) \in C([0,1]; H)$ by

$$
\begin{array}{ll}
0 & \text{if} \quad t \in [0, x_{n,j}] \bigcup [x_{n,j+1}, 1], \\
\sum_{\alpha=0}^p (t - x_{n,j} - \gamma_{n,j})^\alpha A^{-p} a_\alpha \chi_{1,n,j}(t) & \text{if} \quad t \in [x_{n,j},\ x_{n,j} + \gamma_{n,j}], \\
A^{-p} S(t - x_{n,j} - \gamma_{n,j} + \epsilon_0 \theta^j t_n/3) \varphi_{\ell(n,j)} & \text{if} \quad t \in [x_{n,j} + \gamma_{n,j},\ x_{n,j} + \Gamma_{n,j}], \\
\sum_{\alpha=0}^p (t - x_{n,j} - \Gamma_{n,j})^\alpha A^{-p} b_\alpha \chi_{2,n,j}(t) & \text{if} \quad t \in [x_{n,j} + \Gamma_{n,j},\ x_{n,j+1}]
\end{array}
$$

where

$a_\alpha = (\alpha!)^{-1}(-A)^\alpha S(\epsilon_0 3^{-1} \theta^j t_n) \varphi_{\ell(n,j)}$ and $b_\alpha = (\alpha!)^{-1}(-A)^\alpha S((1+\epsilon_0)3^{-1}\theta^j t_n)\varphi_{n,j}$.

REMARK 2. 1) a_α and b_α are α order's coefficients of Taylor expansion of the functions $S(s)\varphi_{n,j}$ at $s = \epsilon_0 \theta^j t_n/3$ and $s = (1 + \epsilon_0)\theta^j t_n/3$ respectively.

2) From the constructive method of the function $f^p_{n,j}$ we see

$(Supp\ f^p_{n_1,j_1}) \cap (Supp\ f^p_{n_2,j_2}) = \emptyset$ if $(n_1, j_1) \neq (n_2, j_2)$.

3) $f^p_{n,j} \in C^p([0,1]; D(A^\infty))$ and it is piecewise sufficiently smooth at $t \in [0,1]$.

LEMMA 3. *Let q and k be nonnegative integers such that $q \leq p$. Then we have*

$$| (d/dt)^q A^k f^p_{n,j}(t) |_H \leq Const \lambda^{q+k-p}_{n,j}.$$

$$(d/dt)(d/dt)^q A^k f^p_{n,j}(t) \in L^2(0,1; H).$$

Proof. We first shall show the former.

Let $t \in [x_{n,j}, x_{n,j} + \gamma_{n,j}]$. From the definition of $\chi_{1,n,j}$ and Lemma 1 it follows

$$| (d/ds)^\beta \chi_{1,n,j} | \leq Const/\gamma^\beta_{n,j} \leq C\lambda^\beta_{\ell(n,j)}. \tag{2.1}$$

If $\beta \leq \alpha$ we have

(2.2) $$| (d/dt)^{\beta}(t - x_{n,j} - \gamma_{n,j})^{\alpha} | \leq Const\gamma_{n,j}^{\alpha-\beta} \leq C\lambda_{\ell(n,j)}^{\beta-\alpha}.$$

From the semigroup properties we see

(2.3). $$| A^k S(s)\varphi_{n,j} |_H \leq Const\lambda_{\ell(n,j)}^{k} exp(-s\lambda_{\ell(n,j)})$$

Combining (2.1),(2.2) and (2.3) we get

(2.4) $$| (d/dt)^q A^k f_{n,j}^p |_H \leq Const\lambda_{\ell(n,j)}^{k-p} exp(-\gamma_{n,j}\lambda_{\ell(n,j)}) \sum_{\alpha=0}^{p} \sum_{\beta=0}^{q\wedge\alpha} \lambda_{\ell(n,j)}^{\beta-\alpha} \lambda_{\ell(n,j)}^{q-\beta}$$

$$\leq Const\lambda_{\ell(n,j)}^{-p+q+k}.$$

Using the similar method to the above, for $t \in [x_{n,j} + \Gamma_{n,j},\ x_{n,j+1}]$, we also get the same estimate as the above.

For $t \in [x_{n,j} + \gamma_{n,j},\ x_{n,j} + \Gamma_{n,j}]$, from (2.3), we also get the same estimate as (2.4). Then the former is proved.

Next we shall show the latter.

If $q+1$ is smaller than p, from the above, it is trivial. We suppose $q = p$. If $t \in (x_{n,j} + \gamma_{n,j}, x_{n,j} + \Gamma_{n,j})$ it follows

$$| (d/dt)(d/dt)^p A^k f_{n,j}^p(t) |_H \leq Const\ \lambda_{\ell(n,j)}^{k+1}.$$

If $t \in (x_{n,j}, x_{n,j} + \gamma_{n,j}) \cup (x_{n,j} + \Gamma_{n,j}, x_{n,j+1})$ it follows

$$(d/dt)(d/dt)^q A^k f_{n,j}^p(t) = 0.$$

Then the latter is proved.

Let b_n be a decreasing sequence such that

(2.5) $$\lim_{n\to\infty} b_n = 0, \quad \inf_n n^{1/2} b_n \geq \delta_0 > 0.$$

From 2) of Remark 2 we know that there exists $\sum_{n=1}^{\infty} \sum_{j=1}^{n} f_{n,j}^p(t) b_n$. Thus we denote the above function by $f^p(t)$.

LEMMA 4. *The function $f^p(\cdot)$ holds the following properties:*

1) $f^p \in C^q([0,1]; D(A^k)) \cap C^p((0,1]; D(A^\infty))$ where $q + k \leq p$.

2) Let δ be any positive small number. This function is piecewise sufficiently smooth on $[\delta, 1]$.

3) $(d/dt + A)^k f^p \in C([0,1]; H)$ and $\lim_{t\to 0}(d/dt + A)^k f^p(t) = 0$ where $k = 0, 1, \cdot\cdot, p$.

4)$(d/dt)(d/dt + A)^p f^p \in L^2_{loc}((0,1]; H)$.

Proof. Combining 2),3) of Remark 2 and lemma 3 and noting (2.5) we get the proof of 1). Since the sum of f^p is finite on $[\delta, 1]$, from 3) of Remark 2, the proof of 2) is complete. From Lemma 3 and (2.5) the proof of 3) is complete. Noting the sum of f^p is finite on $[\delta, 1]$ and Lemma 3 we can prove 4).

LEMMA 5. *Let t be any positive number such that $0 < t \leq 1$. Then there exists*

$$\lim_{\epsilon \searrow 0} \int_\epsilon^t (d/ds)(d/ds + A)^k f^p(s)ds = 0$$

where $k = 0, 1, \cdot\cdot, p$.

Proof. From 2) and 3) of Lemma 4 it is esay to prove this lemma.

LEMMA 6.

$$| A \int_{t_n/2}^{t_n} S(t_n - s)A^p f^p(s)ds |_H \geq \delta n^{1/2} b_n$$

where δ is a positive constant independent of n.

Proof. From the definition of f^p we have $f^p = \sum_{j=1}^n f^p_{n,j} b_n$ on $[t_n/2, t_n]$. We put

$$\int_{x_{n,j}}^{x_{n,j+1}} AS(t_n - s)A^p f^p_{n,j} ds =$$

$$\left(\int_{x_{n,j}}^{x_{n,j}+\gamma_{n,j}} + \int_{x_{n,j}+\gamma_{n,j}}^{x_{n,j}+\Gamma_{n,j}} + \int_{x_{n,j}+\Gamma_{n,j}}^{x_{n,j+1}}\right)\{AS(t_n - s)A^p f^p_{n,j}(s)\}ds$$

$$= I_1 + I_2 + I_3.$$

We first shall estimate I_1. From the definition of $f^p_{n,j}$ on $[x_{n,j}, x_{n,j} + \gamma_{n,j}]$ and semigroup properties we have

$$| AS(t_n - s)A^p f^p_{n,j} |_H$$

$$\leq \sum_{\alpha=0}^{p} 1/(\alpha!) \mid s - x_{n,j} - \gamma_{n,j} \mid^\alpha \lambda_{n,j}^{\alpha+1} exp(-(t_n - s + \epsilon_0 \theta^j t_n/3)\lambda_{n,j}).$$

Since

$$s - x_{n,j} \geq \lambda_{n,j} \quad \text{and} \gamma_{n,j}\lambda_{\ell(n,j)} \leq 1/N$$

we see

$$| I_1 |_H \leq \sum_{\alpha=0}^{p} Const(\gamma_{n,j})^{\alpha+1}\lambda_{\ell(n,j)}^{\alpha+1} \leq Const/N. \tag{2.6}$$

where Const is a constant independent of n, j and N. Using the similar method to the above we get

$$| I_3 |_H \leq Const/N. \tag{2.7}$$

Let us estimete I_2. Using the semigroup properties we get

$$AS(t_n - s)A^p f^p_{n,j} = exp(-(t_n - x_{n,j} + (\epsilon_0 - 1/N)\theta^j t_n/3))\lambda_{n,j}))\lambda_{n,j}\varphi_{n,j}.$$

Since $t_n - x_{n,j} = (1 + 2/N)(1 - \theta)^{-1}\theta^j t_n/3$, from lemma 2 and the above equality we have

$$| I_2 |_H \geq (1 - \epsilon_0)exp(-\delta_1)/3$$

where $\delta_1 = (1-\epsilon_0)\{1/3(1+2/N)(1-\theta)^{-1}+(\epsilon_0-1/N)\}$. Then combining (2.6),(2.7) and the above inequality and noting N is a sufficiently large number there exists a constant δ_0 such that

$$| I_1+I_2+I_3 |_H^2 \geq (| I_2 |_H - | I_1 |_H - | I_3 |_H)^2 \geq ((1-\epsilon_0)exp(-\delta_1)-2Const/N)^2 = \delta_0^2.$$

Thus we complete the proof of this lemma.

LEMMA 7. *Let k be a nonnegative integer such that $k \leq p$. Then we get the following equality:*

$$\int_{t/2}^{t} (t-s)^k A^{k+1} S(t-s)(d/dt+A)^p f^p(s)ds$$

$$= -\sum_{q=0}^{k-1} (t/2)^{k-q} A^{k-q} S(t/2)(d/ds+A)^{p-q-1} A^{j+1} f^p(t/2) C_q$$

$$+C_k \int_{t/2}^{t} S(t-s)(d/ds+A)^{p-k} A^{k+1} f^p(s)ds$$

where $C_q = k!/(k-q)!$.

Proof. Using the integration by parts we get the following recurrence formula for q.

$$\int_{t/2}^{t} (t-s)^{k-q} A^{k+1} S(t-s)(d/ds+A)^{p-q} f^p(s)ds$$

$$= -(t/2)^{k-q} A^{k+1} S(t/2)(d/ds+A)^{k-q-1} f^p(t/2)$$

$$+(k-q) \int_{t/2}^{t} (t-s)^{k-q-1} A^{k+1} S(t-s)(d/ds+A)^{p-q-1} f^p(s)ds.$$

Solving the above recurrence formula we get the proof of this lemma.

LEMMA 8. *We get the following inequality:*

$$\limsup_{t \searrow 0} \left| \int_{t/2}^{t} (t-s)^p A^p S(t-s) d/ds (d/ds+A)^p f^p(s)ds \right|_H > 0.$$

Proof. From the definition of f^p it follows, for any nonnegative integer α,

$$\text{(2.8)} \qquad ((d/dt)^{\alpha} f^p)(t_n/2) = 0 \quad \text{and} \quad ((d/dt)^{\alpha} f^p)(t_n) = 0.$$

Let p be 0. Using the integration by parts and (2.8) we see

$$\left| \int_{t_n/2}^{t_n} S(t_n-s) d/ds f^0(s)ds \right|_H = \left| -A \int_{t_n/2}^{t_n} S(t_n-s) f^0(s)ds \right|_H .$$

From Lemma 6 it follows the right term of the above equation is uniformly positive about n.

Let p be larger than 1. Then from the integration by parts and (2.8) we have

$$\int_{t_n/2}^{t_n} (t_n - s)^p A^p S(t_n - s) d/ds(d/ds + A)^p f^p(s) ds$$

$$= p \int_{t_n/2}^{t_n} (t_n - s)^{p-1} A^p S(t_n - s)(d/ds + A)^p f^p(s) ds$$

$$- \int_{t_n/2}^{t_n} (t_n - s)^p A^{p+1} S(t_n - s)(d/ds + A)^p f^p(s) ds = I_1 + I_2.$$

From Lemma 7 and (2.8) we get

$$I_1 = Const \int_{t_n/2}^{t_n} S(t_n - s)(d/ds + A) A^p f^p(s) ds.$$

On the other hand from the integreation by parts it follows

$$\int_{t_n/2}^{t_n} S(t_n - s)(d/ds + A) A^p f^p(s) ds = 0.$$

Then $I_1 = 0$.

Combining Lemma 6 we obtain $\mid I_2 \mid_H \geq \delta_0$. The proof is complte.

LEMMA 9. *Let k be a nonnegative integer smaller than $p - 1$. Then it follows*

$$\lim_{t \searrow 0} \mid \int_{t/2}^{t} (t - s)^k A^k S(t - s) d/ds(d/ds + A)^p f^p(s) ds \mid_H = 0.$$

Proof. From the integreation by parts we get

$$\int_{t/2}^{t} (t-s)^k A^k S(t-s) d/ds(d/ds+A)^p f^p(s) ds = -(t/2)^k A^k S(t/2)(d/ds+A)^p f^p(t/2)$$

$$+k \int_{t/2}^{t} (t - s)^{k-1} A^k S(t - s)(d/ds + A)^p f^p(s) ds = I_1 + I_2.$$

On the other hand we have the operator norm: $\mid s^k A^k S(s) \mid_{H \to H} \geq Const$. Combining 3) of Lemma 4 and the above result we obtain $lim_{t \searrow 0} I_1 = 0$. From Lemma 7 and 3) of Lemma 4 we get $\lim_{t \searrow 0} I_2 = 0$. Thus the proof is complete.

3. Proof of Theorem.

We take a function f defined on $[0,1]$ such that

$$f(t) = (d/dt)(d/dt + A)^p f^p(t).$$

From then 4) of Lemma 4, Lemma 5, Lemma 8 and Lemma 9 we get

$$f \in F_{p-1} \quad \text{and} \quad f \notin F_p.$$

Combining Proposition 6.9 in [3] and the above result we obtain the proof of Theorem is complete.

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