数理解析研究所講究録
901巻 1995年 8-13

Metrizability of spaces having certain k-networks

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As is well-known, each of the following properties implies that X is metrizable.

(A) X is a paracompact developable space (R. H. Bing [2]).

(B) X is a paracompact space having a σ-locally countable base (V. V. Fedorcuk [9]).

(C) X has a σ-hereditarily closure-preserving base (D. Burke, R. Engelking and D. Lutzer [5]).

(D) X is a paracompact M-space, and a σ-space (A. Okuyama [15]; the paracompactness can be omitted; see F. Siwiec and J. Nagata [16]).

(E) X is an M-space having a point-countable base (V. V. Filippov [10]).

In terms of these properties, we give some metrization theorems by means of certain k-networks, or generalizations of M-spaces, etc.

We assume that spaces are *regular and* T_1.

Definitions. (1) A cover $\mathcal{C}$ of a space a *k-network* if, whenever $K \subset U$ with K compact and U open in X, then $K \subset \cup \mathcal{C}' \subset U$ for some finite $\mathcal{C}' \subset \mathcal{C}$. If the K is a single point, then such a cover is called network (or net). Recall that a space is an $\aleph$-space (resp. σ-space) if it has a σ-locally finite k-network (resp. network).

(2) A space is *countably bi-quasi-k* [14] if, whenever $\{F_n: n \in N\}$ is a decreasing sequence with $x \in \overline{F_n}$, there exists a decreasing sequence $\{A_n: n \in N\}$ such that $x \in \overline{A_n \cap F_n}$ for each $n \in N$, and if $x_n \in A_n$ for each $n \in N$, then the sequence $\{x_n: n \in N\}$ has a cluster point in $\cap\{A_n: n \in N\}$. Recall that a space X is a q-space if each point has a sequence $\{V_n: n \in N\}$ of nbds such that if $x_n \in V_n$, the sequence $\{x_n: n \in N\}$ has a cluster point in X. Every q-space is a countably bi-quasi-k-space. Every countably bi-quasi-k-space is precisely the countably bi-quotient image of an M-space; see [14].

(3) A space X is a *monotonic wΔ-space* (simply, mwΔ-space) [18] (resp. *monotonic developable* [7] (equivalently, space having a base of countable order in the sense of Arhangel'skii [1])), if there exists a sequence $(\mathcal{B}_n)$ of bases for X such that any *decreasing* sequence $\{B_n: n \in N\}$ with $B_n \in \mathcal{B}_n$ satisfies (*):

(*) If $x \in \cap B_n$, and $x_n \in B_n$, then the sequence $\{x_n: n \in N\}$ has a cluster point in X (resp. the cluster point x).

If the sequence $\{B_n: n \in N\}$ with $B_n \in \mathcal{B}_n$ is not necessarily decreasing, then such a space is called a wΔ-space; developable space respectively.

Every wΔ-space or every (monotonic) p-space [7] is an mwΔ-space. Every mwΔ-space is a q-space, hence countably bi-quasi-k. Among θ-refinable (= submetacompact) spaces, mwΔ-spaces (resp. monotonic developable spaces) are wΔ-spaces (resp. developable spaces); see [18] (resp. [7]).

The lemma below holds by means of [13; Proposition 3.2], [17; Section 4], [4; Theorem 4.1], and [18], etc. Here, we note that, in a space having a σ-locally countable k-network, each point is a G_δ-set. So, every countably bi-quasi-k-space with a σ-locally countable k-network is countably bi-k by [14; Theorem 7.3].

Lemma. (1) Suppose that X is a k-space; a normal space in which every closed countably compact set is compact; or each point of X is a G_δ-set. Then (i) and (ii) below hold.

(i) X has a point-countable base if and only if X is a countably bi-quasi-k-space with a point-countable k-network.

(ii) X is a nomotonically developable space with a point-countable base if and only if X is an mwΔ-space with a point-countable k-network (cf.[18]).

(2) (i) A space has a σ-locally countable base if and only if it is a countably bi-quasi-k-space with a σ-locally countable k-network.

(ii) A space is a nomotonically developable space with a σ-locally countable base if and only if it is an mwΔ-space with a σ-locally countable k-network.

Metrization Theorem The following are equivalent.

(a) X is metrizable,

(b) X is a paracompact M-space having a point-countable k-network,

(c) X is an M-space, and a k-space having a point-countable k-network,

(d) X is an M-space having a point-countable k-network, and having a σ-locally countable *network*.

(e) X is an M-space having a σ-locally countable k-network,

(f) X is a paracompact, countably bi-quasi-k-space having a σ-locally countable k-network,

(g) X is a countably bi-quasi-k-space having a point-countable k-network, and having a σ-closure preserving k-network; cf. [12].

(h) X is an mwΔ-space having a σ-closure preserving k-network; see [18].

(i) X is a countably bi-quasi-k-space having a σ-hereditarily closure preserving k-network.

Remark. In the previous theorem, it is possible to replace " countably bi-quasi-k-space " by " countably bi-quasi-k-space, but the sequence $\{x_n : n \in N\}$ has a cluster point in X (instead of $\cap\{A_n : n \in N\}$) in the definition of countably bi-quasi-k-spaces ".

We see that each condition in (b) ~ (h) of Metrization Theorem is essential by means of the following examples.

Examples. (1) Not every countably compact (resp. countably compact, first countable) space having a point-countable k-network (resp. locally countable *network*) is a σ-space.

(2) Not every Čech-complete (resp. metacompact developable) space having a σ-locally countable base is a σ-space [8] (resp. metrizable; cf. [11]).

(3) Not every first countable, Lindelöf space having a σ-closure preserving base is developable [6].

But, the following holds by means of [3] and [18], etc.

Proposition. (1) Every θ-refinable space X is developable if (a) or (b) below holds.

(a) X is an mwΔ-space having a point-countable k-network, or having a σ-locally countable *network*.

(b) X is a countably bi-quasi-k-space having a σ-locally countable k-network.

(2) Every mwΔ-space which is the quotient compact image (resp. quotient s-image) of a metric space is developable (resp. monotonically developable); see [18].

We note that every mwΔ-space having a σ-locally finite *network* (resp. σ-locally countable k-network) is developable (resp. monotonically developable; cf. [18].

In view of the above results, we have the following questions.

Questions. (1) Every wΔ-space having a σ-locally countable k-network (resp. σ-locally countable *network*) is developable (resp. monotonically developable) ?

(2) Every wΔ-space which is the quotient countable-to-one image of a metric space is developable ?

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