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MODULAR FORMS ASSOCIATED WITH THE MONSTER MODULE

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1. Introduction

In Harada-Lang [4], we associated to each irreducible character χ of the monster simple group $\mathbb{M}$ a modular function $t_\chi(z)$, called in [4], the McKay-Thompson series for χ. $t_\chi(z)$ is a weighted average of all McKay-Thompson series $t_g(z)$ for the element g of $\mathbb{M}$ as g ranges over $\mathbb{M}$:

$$t_\chi(z) = \frac{1}{|\mathbb{M}|} \sum_{g \in \mathbb{M}} \chi(g) t_g(z).$$

If Γ_χ is the invariance subgroup of $t_\chi(z)$, then we showed

$$\Gamma_\chi = \Gamma_0(N_\chi) = \bigcap_{g \in \mathbb{M}} \Gamma_g$$

where g ranges over all the elements of $\mathbb{M}$ such that $\chi(g) \neq 0$ and

$$N_\chi = \mathrm{lcm}\{n_g h_g : \text{for all}\, g \in \ \mathbb{M}\, \text{with}\, \chi(g) \neq 0\}.$$

As shown in Conway-Norton [1], the invariance group Γ_g of $t_g(z)$ is a certain subgroup of index h of the conjugate by

$$\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$$

of

$$\Gamma_0(\frac{n}{h}) + e, f, \cdots$$

This is a preliminary version. A full version with a table will be published elsewhere.

where e, f, etc. denote the Atkin-Lehner involutions. In [1], such a conjugate is denoted by

$$n|h + e, f, \cdots.$$

The numbers n, h depend on g, hence our notation n_g, h_g. Obviously every $t_\chi(z)$ is invariant by

$$\bigcap_{g \in \mathbb{M}} \Gamma_g = \Gamma_0(N_0)$$

where $N_0 = 2^6 3^3 5^2 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \sim 10^{21}$. The level N_χ can be very large or relatively small. For example,

$$N_{\chi_1} = N_0, \; N_{\chi_{166}} = 2^6 3^3 7 = 4032$$

where $\chi_1 = 1$ is the trivial character and the character numbering such as χ_{166} is taken from the Atlas. In this paper, we will investigate the relation between $t_\chi(z)$ and the generating functions of the highest weight vectors (also called singular vectors, primary fields or lowest weight vectors.)

2. THE MONSTER MODULE AS A Vir MODULE

The monster module $\mathbb{V}$ is constructed in Frenkel-Lepowsky-Meurman [3] as a vertex operator algebra and is denoted by $\mathbb{V}^\natural$ there. Let V be a vertex operator algebra. Then V possesses two distinguished elements 1 and ω, called the vacuum and the conformal vector (or the Virasoro element) of V, respectively.

If $Y(\omega, z) = \sum \omega_n z^{-n-1}$ is the vertex operator corresponding to the conformal vector ω and if we set $L(n) = \omega_{n+1}$ for $n \in \mathbb{Z}$, then $L(n)$ satisfies the commutation relation:

$$[L(n), L(m)] = (n-m)L_{n+m} + \frac{1}{12}(n^3 - n)c\delta_{n+m,0}$$

where c is a constant called the central charge of V. For the monster module $\mathbb{V}$, $c = 24$. c is also called the rank of the vertex operator algebra V.

Let $\mathcal{L}$ be the Lie algebra generated by $L(n)$, $n \in \mathbb{Z}$. $\mathcal{L}$ is denoted by Vir else where. The subalgebras $\mathcal{L}^+$ and $\mathcal{L}^-$ are generated by $L(n), n \in \mathbb{Z}^+$ and $L(n), n \in \mathbb{Z}^-$, respectively. It is known that $\mathbb{V}$ possesses a positive definite invariant bilinear form and so $\mathbb{V}$ is completely reducible as an $\mathcal{L}$ module and is a sum of highest weight modules.

Let $M(h,c)$ be the Verma module of the Virasoro algebra of central charge c generated by the highest weight vector v of height h : i.e.

$$M(h,c) = \mathcal{L}v,\ \mathcal{L}^+v = 0,\ L(0)v = hv.$$

The module structure of $M(h,c)$ has been determined by Feigin-Fuchs [2]. We will use their results to determine the module structure of $\mathbb{V}$ as an $\mathcal{L}$ module. Feigin-Fuchs showed that every submodule of $M(h,c)$ is a sum of submodules that are also Verma modules. Therefore, the knowledge of all embeddings among Verma modules gives all submodules of a given Verma module. The main theorem of Feigin-Fuchs states that there are six types of embeddings of the Verma modules into other Verma modules. Let

$$\begin{cases} p\alpha - q\beta = m \\ c = 24 = \frac{(3p-2q)(3q-2p)}{pq} \\ h = \frac{m^2-(p-q)^2}{4pq} \end{cases} \tag{1}$$

where p, q and m are complex variables. We next solve for integers α and β. Let

$$\epsilon = \frac{-11 \pm i\sqrt{23}}{2}, \quad \bar{\epsilon} = \frac{-11 \mp i\sqrt{23}}{2}$$

We compute

$$\epsilon\bar{\epsilon} = 1,\ \epsilon + \bar{\epsilon} = \frac{-11}{6},\ \epsilon^2 + \bar{\epsilon}^2 = \frac{49}{36}.$$

Using the second equality of (1), we obtain

$$(p\alpha - q\beta)^2 = m^2 = 4pq + (q-p)^2,$$

which may be rewritten as

$$(\alpha - \epsilon\beta)^2 = 4\epsilon h + (\epsilon - 1)^2.$$

We therefore obtain two equations :

$$\alpha^2 - 2\epsilon\alpha\beta + \epsilon^2\beta^2 = 4\epsilon h + (\epsilon - 1)^2,$$

and

$$\alpha^2 - 2\bar{\epsilon}\alpha\beta + \bar{\epsilon}^2\beta^2 = 4\bar{\epsilon} h + (\bar{\epsilon} - 1)^2.$$

Taking the sum of them, we get

$$72\alpha^2 + 132\alpha\beta + 49\beta^2 = -264h + 253.$$

By subtracting one from the other, we get

$$-12\alpha\beta - 11\beta^2 = 24h - 23.$$

Therefore

$$\alpha^2 - \beta^2 = 0,$$

or $\alpha = \pm\beta$. Setting $\alpha = \delta\beta$ with $\delta = \pm 1$, we have

$$\beta^2 = \frac{24h - 1}{11 - 12\delta}.$$

If $h = 0$, then we must have $\delta = 1$ and so $\alpha = \beta = \pm 1$. In particular, $\alpha\beta = 1 > 0$. On the other hand, if $h \in \mathbb{Z}^+$, then $\delta = -1$ and so $\alpha = -\beta = \pm 1$, and hence $\alpha\beta = -1 < 0$. Using the results of Feigin-Fuchs [2], we conclude (which must be well known to experts) :

Theorem. *$M(0,24)$ has a unique submodule, which is isomorphic to $M(1,24)$. For all positive integers h, $M(h,24)$ is irreducible.*

Let $L(c,h)$ be the unique irreducible highest weight $\mathcal{L}$-module of central charge c and height h. Then

Corollary. *We have*

(1). $L(0,24) = M(0,24)/M(1,24)$, *and,*

(2). $L(h,24) = M(h,24)$ *if* $h \in \mathbb{Z}^+$.

Let us now express the monster module $\mathbb{V}$ as a sum of $L(h,24)$'s as follows

$$\mathbb{V} = \sum_{h=0}^{\infty} s_h L(h,24).$$

Then s_h is the number of linearly independent singular vectors v_h of height h, hence $v_h \in \mathbb{V}_h$. Since the Virasoro algebra $\mathcal{L}$ commutes with the action of the monster $\mathbb{M}$, we can actually split s_h into the sum of s_h^k where the index k corresponds to the irreducible character χ_k. More precisely, let

$$\mathbb{V}_h^k = c_{hk}\chi_k$$

where c_{hk} is the multiplicity of χ_k in $\mathbb{V}_h$ and

$$\mathbb{V}^k = \coprod_{h=0}^{\infty} \mathbb{V}_h^k.$$

Thus $\mathbb{V}^k$ is an $\mathbb{M}$ submodule of $\mathbb{V}$ consisting entirely of irreducible submodules isomorphic to χ_k and $\mathbb{V}_h^k$ is an $\mathbb{M}$ submodule of $\mathbb{V}^k$ of height h. We also define

$$W_h^k = \mathcal{L}\Big(\coprod_{0\le i<h} \mathbb{V}_i^k\Big) \cap \mathbb{V}_h^k,$$

which is an $\mathbb{M}$ submodule of $\mathbb{V}_h^k$ that is generated by elements of lower heights. Let

$$s_h^k = \dim \mathbb{V}_h^k / W_h^k.$$

Then s_h^k is the number of linearly independent singular vectors in $\mathbb{V}_h^k$. Obviously

$$s_h = \sum_{k=1}^{194} s_h^k.$$

For a graded module $X = \sum_{h \in \mathbb{Z}} X_h$, the character of X (or the partition function of X) is defined to be a formal sum

$$\mathrm{char}(X) = \sum_{h \in \mathbb{Z}} \dim X_h x^h.$$

Using this notation, we have, as is well known,

$$\mathrm{char} M(h, c) = x^h \sum_{n \geq 0} p(n) x^n$$

where $p(n)$ is the partition function of n. For convenience, set $p(0) = 1$, and $p(n) = 0$ if $n \in \mathbb{Z}^-$. Let us consider the $\mathcal{L}$ submodule generated by the vacuum 1. We have $V_1 = 0$ but the height 1 component of $M(0, 24)$ is nonzero, we conclude that

$$\mathcal{L} \cdot 1 \simeq M(0, 24)/M(1, 24).$$

Hence

$$\mathrm{char}(\mathcal{L} \cdot 1) = \sum_{n \geq 0} p(n) x^n - x \sum_{n \geq 0} p(n) x^n = \sum_{n \geq 0} (p(n) - p(n-1)) x^n.$$

Writing

$$\mathrm{char}(\mathcal{L} \cdot 1) = \sum_{h \geq 0} a_{h1} x^h,$$

we get a partial list :

h	0	2	3	4	5	6	7	8	9	10	11
a_{h1}	1	1	1	2	2	4	4	7	8	12	14

In [4], we had a partial list of c_{h1} where c_{h1} is the multiplicity of the trivial character χ_1 occuring in $\mathbb{V}_h$.

h	0	2	3	4	5	6	7	8	9	10	11
c_{h1}	1	1	1	2	2	4	4	7	8	12	14

The coincidence $c_{h1} = a_{h1}$ stops there and we have

h	12
a_{h1}	21
c_{h1}	22

This means $s^1_{12} = 1$, namely, $\mathbb{V}^1_{12}$ contains a singular vector, while $\mathbb{V}^1_h$, $0 < h \leq 11$, do not. The number d of linearly independent singular vectors occuring in $\mathbb{V}^1_h$ $(0 \leq h \leq 30)$ is as follows

h	12	16	18	20	22	24	26	27	28	29	30
d	1	1	1	1	1	3	2	1	4	2	6

We are now lead to consider its generating function for each k, $1 \leq k \leq 194$. Define

$$G^k(x) = \sum_{h \geq 0} s^k_h x^h.$$

The character of $\mathbb{V}^k$ is

$$\text{char}(\mathbb{V}^k) = \sum_{h \geq 0} c^k_h (\text{deg}\chi_k) x^h = x\text{deg}\chi_k t_\chi(x)$$

where $t_\chi(z)$ is the McKay-Thompson series for the irreducible character χ. On the other hand, using the expression

$$\mathbb{V}^k = \sum_{h \geq 0} s^k_h L(h, 24),$$

we obtain

$$\mathrm{char}(\mathbb{V}^k) = \sum_{h\geq 0} s_h^k \mathrm{char} L(h,24).$$

Suppose $k > 0$. Then $s_0^k = 0$ and so

$$\mathrm{char}(\mathbb{V}^k) = \sum_{h\geq 1} s_h^k x^h \sum_{n\geq 0} p(n)x^n.$$

On the other hand if $k = 1$, then $L(0,24)$ occurs only once as a constitient of $\mathbb{V}^1$. Therefore

$$\mathrm{char}(\mathbb{V}^1) = (1 - x + \sum_{h\geq 2} s_h^1 x^h) \sum_{n\geq 0} p(n)x^n.$$

Using the Dedekind eta-function and replacing x by $q = e^{2\pi iz}$, we obtain, by setting $s_1^1 = -1$ for convenience,

$$\deg\chi_k t_{\chi_k}(q) = \frac{q^{-1}(\sum_{h\geq 0} s_h^k q^h) q^{\frac{1}{24}}}{\eta(q)}.$$

Hence

$$\deg\chi_k t_{\chi_k}(q)\eta(q) = q^{-\frac{23}{24}} \sum_{h\geq 0} s_h^k q^h,$$

which implies

$$q^{-\frac{23}{24}} G^k(q) = \deg\chi_k t_{\chi_k}(q)\eta(q)$$

where as defined before $G^k(q)$ is the generating function of the singular vectors in $\mathbb{V}^k$. Writing $G^k = G^\chi$ in general, we obtain :

Theorem. *$q^{-\frac{23}{24}} G^\chi(q)$ is a meromorphic modular form of weight $\frac{1}{2}$ and level N_χ.*

Corollary. *$q^{-\frac{23}{24}} G^\chi(q)\eta(q)^{23}$ is a holomorphic modular function of weight 12 and level N_χ.*

REFERENCES

1. Conway, J. H., Norton, S. P., Monstrous Moonshine, *Bull. London Math. Soc.***11** (1979), 308-338.
2. Feigin, B.L., Fuchs, D.B., Verma Modules over the Virasoro Algebras, *Lecture Notes Math.,* 1060 (1984).
3. Frenkel, I., Lepowsky, J., Meurman, A., Vertex Operator Algebras and the Monster, *Academic Press, Inc.* (1988).
4. Harada, K., Lang, M.L., The McKay-Thompson Series associated with the Irreducible Characters of the Monster, *to appear.*

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