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ON SOME ILL-POSED ESTIMATE FOR DEGENERATE ELLIPTIC EQUATIONS OF MONGE-AMPÈRE TYPE WITH TWO VARIABLES

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Abstract

We shall prove an estimate similar to the Hadamard's three circles theorem for solutions of the Cauchy problem for degenerate elliptic equations of Monge-Ampère type with two variables.

1 Introduction

This paper concerns Cauchy problem for degenerate nonlinear elliptic equations of the form

$$\partial_x^2 u \cdot \partial_y^2 u - (\partial_x \partial_y u)^2 + g(x, y, u) = f \tag{1.1}$$

where $f \geq 0$. The degeneracy arises through the vanishing of the function f. In N-dimensional space, the more general form of (1.1) is as follows:

$$det[(\partial_{x_i} \partial_{y_j} u)_{i,j=1}^{N}] + g(x_1, ..., x_N, u, \nabla u) \geq 0. \tag{1.2}$$

The existence and the regularity of solutions of the boundary value problem for (1.2) was studied by several authors(see e.g., [3],[6]).When $N = 2$ and the equation is not degenerate, there is the famous book [5] of Pogorelov, where the boundary value problem is mainly discussed.

We explain briefly the Cauchy problem of elliptic equations. Let L be a linear elliptic operator of second order, and Ω be a domain in N-dimensional space. Let Γ be an initial surface on $\partial\Omega$, and $\mathbf{n}$ be the outer normal of $\partial\Omega$. The Cauchy problem is to find a solution u such that $Lu = 0$ in Ω and $u = \varphi$, $\partial_{\mathbf{n}} u = \psi$ on Γ, where φ and ψ are any two given function. It is well-known that such a problem is ill-posed. But the uniqueness holds, that is, u vanishes identically, if $\varphi = \psi = 0$ on Γ. The uniquness for more general linear equations is precisely discussed in [2], where the method is either Holmgren's or Carleman's one. Let Ω'

be a bounded subdomain of Ω such that $\overline{\Omega'} \subset \Omega \cup \Gamma$. Then it holds too that there are two constants C and α with $0 < \alpha < 1$ such that

$$\|u\|_2 \leq C(\|u\|_1)^{\alpha}(\|u\|_3)^{1-\alpha} \tag{1.3}$$

where $\| \ \|_i$ $(i = 1, 2, 3)$ are some norms on Γ, Ω' and Ω, respectively. This inequality is of type with Hadamard's three circles theorem. The quantity $\|u\|_1$ is that of φ and ψ. Hadamard's three circles theorem for linear elliptic epuations was proved previously by several authors (see e.g., [4]).

Naturally the following question arises : Does (1.3) hold too for solutions of nonlinear elliptic equations of second order? For example, Výborný [7] proved Hadamard's three circles theorem for nonlinear uniformly elliptic operators. His method is to prepare a kind of the maximum principle.

In this paper we consider the Cauchy problem for (1.1) and we prove an inequality similar to (1.3) for solutions of (1.1). Our method is due to Carleman's inequality, which was often used up to now in order to prove the unique continuation property for solutions of elliptic equations with linear principal parts. Recently, Hayasida [1] has proved an inequality as in (1.3) for solutions of a degenerate quasilinear elliptic equation with Carleman's method. Our research is motivated by [1], but the tool in this paper is different from [1] in several points.

2 Results

Let D be a bounded domain in the (x, y)-plane with its boundary ∂D. Let Γ be a connected open subset of ∂D. We assume that $D \subset \{y > 0\}$, $\Gamma \ni O$(the origin) and Γ is of class C^1.

We write for $\rho > 0$, $D_\rho = D \cap \{y < \rho\}$, $\Gamma_\rho = \Gamma \cap \{y < \rho\}$, $l_\rho = D \cap \{y = \rho\}$ (see Figure 1). We define the following definitions :

(H.1) There is a real number a with $0 < a < 1$ such that each l_ρ is an open segment and $|l_\rho| \leq |l_{\rho'}| \leq 1/2$ for any ρ, ρ' with $0 < \rho < \rho' < a$.

If (H.1) is satisfied, let us say often that (H.1) holds for D_a.

(H.2) Under the hypothesis of (H.1), there is a number $c > 0$ and a function $\varphi(x) \in C^2(\{|x| \leq c\})$ such that $\varphi(\pm c) \geq a$, $\{(x, \varphi(x)); |x| \leq c\} \subset \Gamma$ and $\varphi''(x) > 0$ in $\{|x| \leq c\}$.

In (1.1) we assume that the lower order term g has the form

$$g(x, y, z) \leq Kz^2$$

for some positive constant K. So the equation (1.1) becomes

$$(\partial_x \partial_y u)^2 - \partial_x^2 u \cdot \partial_y^2 u \leq Ku^2. \tag{2.1}$$

We denote the norms of $L^\infty(D_\rho)$, $L^\infty(\Gamma_\rho)$ and $L^\infty(l_\rho)$ by $\| \ \|_\rho$, $\langle \ \rangle'_\rho$ and $\langle \ \rangle''_\rho$, respectively.

Our aim is to prove

Theorem 1 *Suppose that* (H.1) *is satisfied. Suppose that* u *belongs to* $C^2(\overline{D_a})$ *and it is a solution of* (2.1) *in* D_a. *Let*

$$\varepsilon = \langle u\rangle_a' + \langle \partial_x u\rangle_a' + \langle \partial_y u\rangle_a' + \langle \partial_x\partial_y u\rangle_a' + \langle \partial_y^2 u\rangle_a',$$

$$M = \langle u\rangle_a'' + \langle \partial_x u\rangle_a'' + \langle \partial_y u\rangle_a'',$$

And let

$$\varepsilon \cdot \max(e^a, e^{\sqrt{2K}a}) \leq M.$$

Then it holds that

$$\|u\|_{\frac{a}{2}} + \|\partial_x u\|_{\frac{a}{2}} \leq C a^{-2} \varepsilon^{\frac{1}{3}} M^{\frac{2}{3}},$$

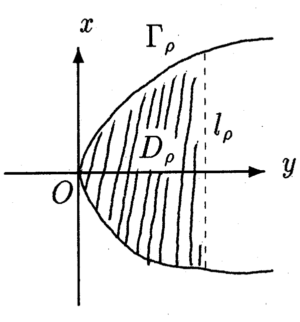
where C *is a positive constant independent of* a, K, ε, M, Γ *and* D.

Remark The inequality (2.1) is invariant under the orthognal transformation of coordinates. So we can generalize the domain by the rotation of D arround the origin.

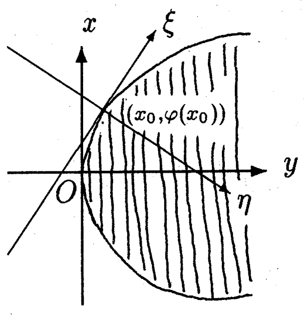
Next we assume (H.2). Let x_0 be a real number such that $0 < |x_0| < c/2$ and $|\varphi'(x_0)| < 1/2$. Around the point $(x_0, \varphi(x_0))$ we take the orthognal transformation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - \varphi(x_0) \end{pmatrix},$$

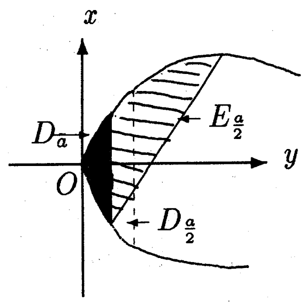
where $\sin\theta = \varphi'(x_0)/\sqrt{1+\varphi'(x_0)^2}$. That is, $\xi(\eta)$-axis is the tangent(normal) line of Γ at $(x_0, \varphi(x_0))$, respectively (see Figure 2). We define for $\rho > 0$, $E_\rho = D \cap \{(\xi,\eta); 0 < \eta < \rho\}$. We look at D as a domain in the new plane with (ξ,η)-coordinates. Under the assumption (H.2), the following is easily verified: There are x_0, a, $\tilde{a}$ such that $0 < |x_0| < c/2$, $|\varphi'(x_0)| < 1/2$, $0 < \tilde{a} < a/2$, $D_{\tilde{a}} \subset D_{\frac{a}{2}} \cap E_{\frac{a}{2}}$, and (H.1) holds for both D_a and E_a(see Figure 3). We denote $\beta = \sin\theta$.



(Figure 1)



(Figure 2)



(Figure 3)

Let $\langle\ \ \rangle_\Gamma$ be the norm in $L^\infty(\Gamma)$. Let F be the intersection of $\overline{D}$ and the minimum convex set containing Γ. We denote by $\|\ \ \|_F$ the norm in $L^\infty(F)$. Under these assumptions we have

Theorem 2 *Let u be the function in Theorem 1. Let*

$$\tilde{\varepsilon} = \langle u\rangle_\Gamma + \langle \partial_x u\rangle_\Gamma + \langle \partial_y u\rangle_\Gamma + \langle \partial_x\partial_y u\rangle_\Gamma + \langle \partial_y^2 u\rangle_\Gamma,$$

$$\tilde{M} = \|u\|_F + \|\partial_x u\|_F + \|\partial_y u\|_F.$$

And let

$$\tilde{\varepsilon}\cdot\max(e^a, e^{\sqrt{2Ka}}) \leq \tilde{M}.$$

Then it holds that

$$\|u\|_{\tilde{a}} + \|\partial_x u\|_{\tilde{a}} + |\beta|\|\partial_y u\|_{\tilde{a}} \leq Ca^{-2}(\tilde{\varepsilon} + |\beta|\langle \partial_x^2 u\rangle_\Gamma)^{\frac{1}{3}}\tilde{M}^{\frac{2}{3}},$$

where C is a positive constant independent of a, $\tilde{a}$, K, ε, M, Γ, D, x_0 and β.

3 Lemmas

We prepare two lemmas. We assume (H.1) and a is the real number in (H.1).

Lemma 1 *Let f belong to $C^1(\overline{D}_a)$ and $f \geq 0$ in D_a. Then it holds that*

$$-\iint_{D_a} \partial_y f dxdy \leq \int_{\Gamma_a} f d\sigma.$$

Proof. Let $0 < \rho < a$ and $l_\rho = (x_1(\rho), x_2(\rho))$. Obviously

$$\partial_\rho\Big(\int_{l_\rho} f(x,\rho)dx\Big) = \int_{l_\rho} \partial_\rho f(x,\rho)dx + f((x_2(\rho),\rho)x_2'(\rho) - f(x_1(\rho),\rho)x_1'(\rho).$$

Noting that $d\sigma = (1 + x_i'(\rho)^2)^{\frac{1}{2}}d\rho$ $(i = 1,2)$, we have

$$\int_0^a [f(x_2(\rho),\rho)x_2'(\rho) - f(x_1(\rho),\rho)x_1'(\rho)]d\rho \leq \int_{\Gamma_a} f d\sigma.$$

Hence

$$-\int_0^a\int_{l_\rho} \partial_\rho f dxdy \leq -\int_0^a \partial_\rho\Big(\int_{l_\rho} f d\sigma\Big)d\rho + \int_{\Gamma_a} f d\sigma = -\int_{l_a} f d\sigma + \int_{\Gamma_a} f d\sigma.$$

This completes the proof. ∎

The following lemma is known to all (see e.g., Lemma 3 in [1]). But we give again its proof.

Lemma 2 *Let $p \geq 1$ and f belong to $C^1(\overline{D}_a)$. Then it holds that for ρ with $0 < \rho < a$*

$$\iint_{D_\rho} |f|^p dxdy \leq 2^p \int_{\Gamma_\rho} |f|^p d\sigma + (2|l_\rho|)^p \iint_{D_\rho} |\partial_x f|^p dxdy.$$

Proof. Let $x_2(y)$ be the function in the proof of Lemma 1. Let (x, y) be in D_ρ. Then from the trivial equality

$$f(x,y) = f(x_2(y), y) + \int_{x_2(y)}^{x} \partial_x f(t,y)dt,$$

we have

$$|f(x,y)|^p \leq 2^{p-1}[|f(x_2(y),y)|^p + (\int_{x}^{x_2(y)} |\partial_x f(t,y)|dt)^p].$$

From Hölder's inequality

$$(\int_{x}^{x_2(y)} |\partial_x f(t,y)|dt)^p \leq |l_y|^{p-1} \int_{l_y} |\partial_x f(t,y)|^p dt.$$

Hence integrating the both sides of the above inequality, we obtain

$$\int_{l_y} |f(x,y)|^p dx \leq 2^{p-1}[|f(x_2(y),y)|^p + |l_y|^p \int_{l_y} |\partial_x f|^p dx].$$

The required inequality follows form this inequality by integration with respect to y. ∎

4 Proof of Theorem 1

We give the proof of Theorem 1 in this section. Let $0 < \rho \leq a$ and u be the function in Theorem 1. We denote by $(\ ,\ )_\rho$ the inner product of $L^2(D_\rho)$.

Let us set $v(x,y) = e^{\lambda y}u(x,y)$ for $\lambda \leq -1$. It is easily verified from (2.1) that

$$(\partial_x\partial_y v)^2 - \partial_x^2 v \cdot \partial_y^2 v - 2\lambda\partial_x v \cdot \partial_x\partial_y v + \lambda^2(\partial_x v)^2 + 2\lambda\partial_y v \cdot \partial_x^2 v - \lambda^2 v\partial_x^2 v \leq Kv^2.$$

From this it follows that for $k > 0$

$$\begin{aligned}(4.1)\quad &((\partial_x\partial_y v)^2, |\partial_x v|^k)_\rho - (\partial_x^2 v\cdot\partial_y^2 v, |\partial_x v|^k)_\rho - 2\lambda(\partial_x v\cdot\partial_x\partial_y v, |\partial_x v|^k)_\rho \\ &+\lambda^2((\partial_x v)^2, |\partial_x v|^k)_\rho + 2\lambda(\partial_y v\cdot\partial_x^2 v, |\partial_x v|^k)_\rho - \lambda^2(v\partial_x^2 v, |\partial_x v|^k)_\rho \leq K(v^2, |\partial_x v|^k)_\rho.\end{aligned}$$

After here let $\mathbf{n}$ be the outer normal of ∂D_ρ. And let $(x, \mathbf{n})$ $((y, \mathbf{n}))$ be the angle between x-axis (y-axis) and $\mathbf{n}$, respectively. By integration by parts we see that

$$\begin{aligned}-(\partial_x^2 v\cdot\partial_y^2 v, |\partial_x v|^k)_\rho &= -\frac{1}{1+k}(\partial_x(|\partial_x v|^k\partial_x v), \partial_y^2 v)_\rho \\ &= -\frac{1}{1+k}\int_{\partial D_\rho} |\partial_x v|^k\partial_x v\cdot\partial_y^2 v\cos(x,\mathbf{n})d\sigma + \frac{1}{1+k}(|\partial_x v|^k\partial_x v, \partial_x\partial_y^2 v)_\rho,\end{aligned}$$

and

$$(|\partial_x v|^k\partial_x v, \partial_x\partial_y^2 v)_\rho = \int_{\partial D_\rho} |\partial_x v|^k\partial_x v\cdot\partial_x\partial_y v\cos(y,\mathbf{n})d\sigma - (1+k)(|\partial_x v|^k, (\partial_x\partial_y v)^2)_\rho.$$

Here the third derivatives of v appear. But it is not necessary to assume three times differentiability of v, if we take an approximating sequence of v. Thus we have

$$\begin{aligned}-(\partial_x^2 v\cdot\partial_y^2 v, |\partial_x v|^k)_\rho &= -\frac{1}{1+k}\int_{\partial D_\rho} |\partial_x v|^k\partial_x v\cdot\partial_y^2 v\cos(x,\mathbf{n})d\sigma \\ &+\frac{1}{1+k}\int_{\partial D_\rho} |\partial_x v|^k\partial_x v\cdot\partial_x\partial_y v\cos(y,\mathbf{n})d\sigma - (|\partial_x v|^k, (\partial_x\partial_y v)^2)_\rho.\end{aligned}$$

Further we have the following equalities:

$$\begin{aligned}(\partial_x v\cdot\partial_x\partial_y v, |\partial_x v|^k)_\rho &= \frac{1}{2+k}(1, \partial_y(|\partial_x v|^{2+k}))_\rho \\ &= \frac{1}{2+k}\int_{\partial D_\rho} |\partial_x v|^{2+k}\cos(y,\mathbf{n})d\sigma,\end{aligned}$$

$$\begin{aligned}(\partial_y v\cdot\partial_x^2 v, |\partial_x v|^k)_\rho &= \frac{1}{1+k}(\partial_y v, \partial_x(|\partial_x v|^k\partial_x v))_\rho \\ &= \frac{1}{1+k}\int_{\partial D_\rho}|\partial_x v|^k\partial_x v\cdot\partial_y v\cos(x,\mathbf{n})d\sigma - \frac{1}{1+k}(\partial_x\partial_y v, |\partial_x v|^k\partial_x v)_\rho \\ &= \frac{1}{1+k}\int_{\partial D_\rho}|\partial_x v|^k\partial_x v\cdot\partial_y v\cos(x,\mathbf{n})d\sigma \\ &\qquad -\frac{1}{(1+k)(2+k)}\int_{\partial D_\rho}|\partial_x v|^{2+k}\cos(y,\mathbf{n})d\sigma,\end{aligned}$$

$$\begin{aligned}(v\partial_x^2 v, |\partial_x v|^k)_\rho &= \frac{1}{1+k}(v, \partial_x(|\partial_x v|^k\partial_x v))_\rho \\ &= \frac{1}{1+k}\int_{\partial D_\rho}|\partial_x v|^k\partial_x v\cdot v\cos(x,\mathbf{n})d\sigma - \frac{1}{1+k}(1, |\partial_x v|^{2+k})_\rho.\end{aligned}$$

Combining the above equalities with (4.1), we obtain

$$\begin{aligned}(4.2)\quad \frac{2+k}{1+k}\lambda^2(1,|\partial_x v|^{2+k})_\rho \;\le\; & \frac{1}{1+k}\Big[\int_{\partial D_\rho}|\partial_x v|^k\partial_x v\cdot\partial_y^2 v\cos(x,\mathbf{n})d\sigma \\ & \qquad -\int_{\partial D_\rho}|\partial_x v|^k\partial_x v\cdot\partial_x\partial_y v\cos(y,\mathbf{n})d\sigma\Big] \\ & + \frac{2}{2+k}\lambda\int_{\partial D_\rho}|\partial_x v|^{2+k}\cos(y,\mathbf{n})d\sigma \\ & - \frac{2}{1+k}\lambda\int_{\partial D_\rho}|\partial_x v|^k\partial_x v\cdot\partial_y v\cos(x,\mathbf{n})d\sigma \\ & + \frac{2}{(1+k)(2+k)}\lambda\int_{\partial D_\rho}|\partial_x v|^{2+k}\cos(y,\mathbf{n})d\sigma \\ & + \frac{1}{1+k}\lambda^2\int_{\partial D_\rho}|\partial_x v|^k\partial_x v\cdot v\cos(x,\mathbf{n})d\sigma \\ & + K(v^2,|\partial_x v|^k)_\rho.\end{aligned}$$

From now on let k be sufficiently large and let us take λ with $2K \le \lambda^2$. Obviously we have the following inequalities

$$\int_{\partial D_\rho}|\partial_x v|^k\partial_x v\cdot\partial_y^2 v\cos(x,\mathbf{n})d\sigma \le \int_{\Gamma_\rho}|\partial_x v|^{1+k}|\partial_y^2 v|d\sigma,$$

$$-\int_{\partial D_\rho}|\partial_x v|^k\partial_x v\cdot\partial_x\partial_y v\cos(y,\mathbf{n})d\sigma \le \int_{\Gamma_\rho}|\partial_x v|^{1+k}|\partial_x\partial_y v|d\sigma - \frac{1}{2+k}\int_{l_\rho}\partial_y(|\partial_x v|^{2+k})d\sigma.$$

By Cauchy-Young's inequality

$$\int_{\Gamma_\rho} |\partial_x v|^{1+k}|\partial_y^2 v|d\sigma \le \frac{1+k}{2+k}\int_{\Gamma_\rho}|\partial_x v|^{2+k}d\sigma + \frac{1}{2+k}\int_{\Gamma_\rho}|\partial_y^2 v|^{2+k}d\sigma,$$

$$\int_{\Gamma_\rho} |\partial_x v|^{1+k}|\partial_x\partial_y v|d\sigma \le \frac{1+k}{2+k}\int_{\Gamma_\rho}|\partial_x v|^{2+k}d\sigma + \frac{1}{2+k}\int_{\Gamma_\rho}|\partial_x\partial_y v|^{2+k}d\sigma,$$

$$\int_{\partial D_\rho} |\partial_x v|^{k}\partial_x v\cdot\partial_y v\cos(x,\mathbf{n})d\sigma \le \frac{1+k}{2+k}\int_{\partial D_\rho}|\partial_x v|^{2+k}d\sigma + \frac{1}{2+k}\int_{\partial D_\rho}|\partial_y v|^{2+k}d\sigma,$$

$$\int_{\partial D_\rho} |\partial_x v|^{k}\partial_x v\cdot v\cos(x,\mathbf{n})d\sigma \le \frac{1+k}{2+k}\int_{\partial D_\rho}|\partial_x v|^{2+k}d\sigma + \frac{1}{2+k}\int_{\partial D_\rho}|v|^{2+k}d\sigma,$$

$$(v^2,|\partial_x v|^k)_\rho \le \frac{k}{2+k}(1,|\partial_x v|^{2+k})_\rho + \frac{2}{2+k}(1,|v|^{2+k})_\rho.$$

Combining these inequalities with (4.2), we obtain

$$\begin{aligned}
\Big(\frac{2+k}{1+k}\lambda^2 - \frac{k}{2+k}K\Big)(1,|\partial_x v|^{2+k})_\rho \;\le\;& \frac{2}{2+k}\int_{\partial D_\rho}|\partial_x v|^{2+k}d\sigma \\
&+ \frac{1}{(1+k)(2+k)}\Big[\int_{\Gamma_\rho}|\partial_y^2 v|^{2+k}d\sigma + \int_{\Gamma_\rho}|\partial_x\partial_y v|^{2+k}d\sigma\Big] \\
&- \frac{1}{(1+k)(2+k)}\int_{l_\rho}\partial_y(|\partial_x v|^{2+k})d\sigma \\
&+ \frac{4}{2+k}|\lambda|\int_{\partial D_\rho}|\partial_x v|^{2+k}d\sigma \\
&+ \frac{2}{(1+k)(2+k)}|\lambda|\Big[\int_{\partial D_\rho}|\partial_x v|^{2+k}d\sigma + \int_{\partial D_\rho}|\partial_y v|^{2+k}d\sigma\Big] \\
&+ \frac{1}{2+k}\lambda^2\int_{\partial D_\rho}|\partial_x v|^{2+k}d\sigma \\
&+ \frac{1}{(1+k)(2+k)}\lambda^2\int_{\partial D_\rho}|v|^{2+k}d\sigma \\
&+ \frac{2K}{2+k}(1,|v|^{2+k})_\rho.
\end{aligned}$$

Since $\lambda^2/2 < (2+k)\lambda^2/(1+k) - kK/(2+k)$ and k is large, this becomes

$$\begin{aligned}
(1,|\partial_x v|^{2+k})_\rho \;\le\;& \int_{\partial D_\rho}|\partial_x v|^{2+k}d\sigma + \int_{\Gamma_\rho}|\partial_y^2 v|^{2+k}d\sigma \\
&+ \int_{\Gamma_\rho}|\partial_x\partial_y v|^{2+k}d\sigma - \frac{2}{\lambda^2(1+k)(2+k)}\int_{l_\rho}\partial_y(|\partial_x v|^{2+k})d\sigma \\
&+ \int_{\partial D_\rho}|\partial_y v|^{2+k}d\sigma + \int_{\partial D_\rho}|v|^{2+k}d\sigma \\
&+ \frac{4K}{\lambda^2(2+k)}(1,|v|^{2+k})_\rho.
\end{aligned} \tag{4.3}$$

From Lemma 2

(4.4) $$(1,|v|^{2+k})_\rho \le 2^{2+k}\int_{\Gamma_\rho}|v|^{2+k}d\sigma + (1,|\partial_x v|^{2+k})_\rho.$$

Hence we have from (4.3)

$$\begin{aligned}(4.5)\qquad (1,|\partial_x v|^{2+k})_\rho &\le 2[\int_{\partial D_\rho}|\partial_x v|^{2+k}d\sigma + \int_{\Gamma_\rho}|\partial_y^2 v|^{2+k}d\sigma \\ &+ \int_{\Gamma_\rho}|\partial_x\partial_y v|^{2+k}d\sigma - \frac{1}{\lambda^2(1+k)(2+k)}\int_{l_\rho}\partial_y(|\partial_x v|^{2+k})d\sigma \\ &+ \int_{\partial D_\rho}|\partial_y v|^{2+k}d\sigma + \int_{\partial D_\rho}|v|^{2+k}d\sigma \\ &+ \frac{2K}{\lambda^2(2+k)}2^{2+k}\int_{\Gamma_\rho}|v|^{2+k}d\sigma].\end{aligned}$$

Now we integrate the both sides of (4.5) with respect to ρ. In general it holds that for any $f \in C^0(\overline{D_a})$, $f \ge 0$ in D_a

$$\int_0^a\int_{\partial D_\rho} f d\sigma d\rho = \int_0^a(\int_{\Gamma_\rho} f d\sigma + \int_{l_\rho} f d\sigma) \le \int_{\Gamma_a} f d\sigma + \int_{l_a} f d\sigma \le \int_{\partial D_a} f d\sigma,$$

$$\int_0^a (1,f)_\rho d\rho = \int_0^a \rho'(1,f)_\rho d\rho = a(1,f)_a - \int_0^a \rho\partial_\rho(1,f)_\rho d\rho = (a-y,f)_a \ge \frac{a}{2}(1,f)_{\frac{a}{2}}.$$

From lemma 1

$$-\int_0^a\int_{l_\rho}\partial_y(|\partial_x v|^{2+k})d\sigma d\rho \le \int_{\Gamma_a}|\partial_x v|^{2+k}d\sigma.$$

From the above (4.5) becomes

$$\begin{aligned}\frac{a}{2}(1,|\partial_x v|^{2+k})_{\frac{a}{2}} &\le 3[\int_{\partial D_a}(|\partial_x v|^{2+k} + |\partial_y v|^{2+k} + |v|^{2+k})d\sigma \\ &+ \int_{\Gamma_a}(|\partial_y^2 v|^{2+k} + |\partial_x\partial_y v|^{2+k})d\sigma + 2^{2+k}\int_{\Gamma_a}|v|^{2+k}d\sigma].\end{aligned}$$

Hence

$$\begin{aligned}(4.6)\ (\int_{D_{\frac{a}{2}}}|\partial_x v|^{2+k}dxdy)^{\frac{1}{2+k}} &\le (6/a)^{\frac{1}{2+k}}[(\int_{\partial D_a}|\partial_x v|^{2+k}d\sigma)^{\frac{1}{2+k}} + (\int_{\partial D_a}|\partial_y v|^{2+k}d\sigma)^{\frac{1}{2+k}} \\ &+ (\int_{\partial D_a}|v|^{2+k}d\sigma)^{\frac{1}{2+k}} + (\int_{\Gamma_a}|\partial_y^2 v|^{2+k}d\sigma)^{\frac{1}{2+k}} \\ &+ (\int_{\Gamma_a}|\partial_x\partial_y v|^{2+k}d\sigma)^{\frac{1}{2+k}} + 2(\int_{\Gamma_a}|v|^{2+k}d\sigma)^{\frac{1}{2+k}}],\end{aligned}$$

where we have used the inequality $(\sum_i a_i)^{\frac{1}{p}} \le \sum_i a_i^{\frac{1}{p}}$ for $p \ge 1$ and $a_i \ge 0$. Letting $k \to \infty$ in (4.5), we obtain

$$\begin{aligned}\|\partial_x v\|_{\frac{a}{2}} &\le \langle\partial_x v\rangle'_a + \langle\partial_y v\rangle'_a + 3\langle v\rangle'_a + \langle\partial_y^2 v\rangle'_a + \langle\partial_x\partial_y v\rangle'_a \\ &+ \langle\partial_x v\rangle''_a + \langle\partial_y v\rangle''_a + \langle v\rangle''_a.\end{aligned}$$

And from (4.4)

$$\|v\|_{\frac{a}{2}} \leq 2\langle v\rangle'_{\frac{a}{2}} + \|\partial_x v\|_{\frac{a}{2}}.$$

Combining these two inequalities, we conclude that

$$\begin{aligned} \|v\|_{\frac{a}{2}} + \|\partial_x v\|_{\frac{a}{2}} &\leq C[\langle v\rangle'_a + \langle \partial_x v\rangle'_a + \langle \partial_y v\rangle'_a + \langle \partial_x\partial_y v\rangle'_a + \langle \partial_y^2 v\rangle'_a \\ &+ \langle v\rangle''_a + \langle \partial_x v\rangle''_a + \langle \partial_y v\rangle''_a], \end{aligned} \tag{4.7}$$

where C is independent of a, K, ε, M, λ, Γ and D.

Here we use the following inequalities:

$$\begin{aligned} &\langle v\rangle'_a \leq \langle u\rangle'_a, \qquad \langle \partial_x v\rangle'_a \leq \langle \partial_x u\rangle'_a, \\ &\langle \partial_y v\rangle'_a \leq \langle \partial_y u\rangle'_a + |\lambda|\langle u\rangle'_a, \\ &\langle \partial_x\partial_y v\rangle'_a \leq \langle \partial_x\partial_y u\rangle'_a + |\lambda|\langle \partial_x u\rangle'_a, \\ &\langle \partial_y^2 v\rangle'_a \leq \langle \partial_y^2 u\rangle'_a + 2|\lambda|\langle \partial_y u\rangle'_a + \lambda^2\langle u\rangle'_a, \\ &\langle v\rangle''_a \leq e^{\lambda a}\langle u\rangle''_a, \qquad \langle \partial_x v\rangle''_a \leq e^{\lambda a}\langle \partial_x u\rangle''_a, \\ &\langle \partial_y v\rangle''_a \leq e^{\lambda a}(\langle \partial_y u\rangle''_a + |\lambda|\langle u\rangle''_a). \end{aligned}$$

Then (4.7) becomes

$$\begin{aligned} e^{\frac{\lambda a}{2}}(\|u\|_{\frac{a}{2}} + \|\partial_x u\|_{\frac{a}{2}}) &\leq C\lambda^2[\langle u\rangle'_a + \langle \partial_x u\rangle'_a + \langle \partial_y u\rangle'_a + \langle \partial_x\partial_y u\rangle'_a \\ &+ \langle \partial_y^2 u\rangle'_a + e^{\lambda a}(\langle u\rangle''_a + \langle \partial_x u\rangle''_a + \langle \partial_y u\rangle''_a)]. \end{aligned}$$

From the definitions of ε and M we can write

$$\|u\|_{\frac{a}{2}} + \|\partial_x u\|_{\frac{a}{2}} \leq C\lambda^2 e^{\frac{|\lambda|a}{2}}(\varepsilon + e^{\lambda a}M).$$

We set

$$\lambda = -\frac{1}{a}\log(\frac{M}{\varepsilon}).$$

Then $\lambda \leq -1$ and $2K \leq \lambda^2$ from our assumption on ε and M. Since $\frac{1}{2}(a|\lambda|/6)^2 \leq e^{\frac{|\lambda|a}{6}}$, we have $\lambda^2 \leq Ca^{-2}e^{\frac{|\lambda|a}{6}}$. Therefore

$$\|u\|_{\frac{a}{2}} + \|\partial_x u\|_{\frac{a}{2}} \leq Ca^{-2}(e^{\frac{2|\lambda|a}{3}}\varepsilon + e^{\frac{\lambda a}{3}}M) = Ca^{-2}[\varepsilon(\frac{M}{\varepsilon})^{\frac{2}{3}} + M(\frac{\varepsilon}{M})^{\frac{1}{3}}].$$

This completes the proof of Theorem 1.

5 Proof of Theorem 2

For simplicity we may assume that $x_0 > 0$. Thus $0 < \varphi'(x_0)$. We write $\alpha = \cos\theta$. Then

$$\frac{2}{\sqrt{5}} < \alpha < 1, \quad 0 < \beta < \frac{1}{\sqrt{5}}.$$

Since

$$\begin{pmatrix} x - x_0 \\ y - \varphi(x_0) \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

we have the following relations :

$$\left\{\begin{array}{l}\partial_\xi u = \alpha\partial_x u + \beta\partial_y u,\\ \partial_\eta u = \alpha\partial_y u - \beta\partial_x u,\\ \partial_\xi\partial_\eta u = -\alpha\beta\partial_x^2 u + (\alpha^2-\beta^2)\partial_x\partial_y u + \alpha\beta\partial_y^2 u,\\ \partial_\eta^2 u = \beta^2\partial_x^2 u - 2\alpha\beta\partial_x\partial_y u + \alpha^2\partial_y^2 u.\end{array}\right. \tag{5.1}$$

We define

$$\varepsilon' = \langle u\rangle_\Gamma + \langle\partial_\xi u\rangle_\Gamma + \langle\partial_\eta u\rangle_\Gamma + \langle\partial_\xi\partial_\eta u\rangle_\Gamma + \langle\partial_\eta^2 u\rangle_\Gamma,$$

$$M' = \|u\|_F + \|\partial_\xi u\|_F + \|\partial_\eta u\|_F.$$

From Theorem 1 it holds that

$$\|u\|_{\tilde{a}} + \|\partial_\xi u\|_{\tilde{a}} \le Ca^{-2}\varepsilon'^{\frac{1}{3}}M'^{\frac{2}{3}}$$

and

$$\|u\|_{\tilde{a}} + \|\partial_x u\|_{\tilde{a}} \le Ca^{-2}\varepsilon^{\frac{1}{3}}M^{\frac{2}{3}},$$

where ε and M are the quantities in Theorem 1. From (5.1)

$$\|u\|_{\tilde{a}} + \|\partial_x u\|_{\tilde{a}} + \beta\|\partial_y u\|_{\tilde{a}} \le \|u\|_{\tilde{a}} + 2\|\partial_x u\|_{\tilde{a}} + \|\partial_\xi u\|_{\tilde{a}}.$$

Thus we have

$$\|u\|_{\tilde{a}} + \|\partial_x u\|_{\tilde{a}} + \beta\|\partial_y u\|_{\tilde{a}} \le Ca^{-2}(\varepsilon+\varepsilon')^{\frac{1}{3}}(M+M')^{\frac{2}{3}}.$$

It is immediately seen from (5.1) that

$$\begin{aligned}\varepsilon' &\le 2[\langle u\rangle_\Gamma + \langle\partial_x u\rangle_\Gamma + \langle\partial_y u\rangle_\Gamma\\ &+ \langle\partial_x\partial_y u\rangle_\Gamma + \langle\partial_y^2 u\rangle_\Gamma] + \beta\langle\partial_x^2 u\rangle_\Gamma,\\ M' &\le 2(\|u\|_F + \|\partial_x u\|_F + \|\partial_y u\|_F).\end{aligned}$$

From these inequlities we finish the proof of Theorem 2.

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